A TEXT BOOK OF HIGHER SECONDARY

Elective Mathematics

[IN ENGLISH]

For Class IX

2659

KESHAB CHANDRA NAG

NAG BOOK HOUSE

This book was taken from the Library of Extension Services Department on the date last stamped. It is returnable within 7 days.

7.8.75

Written strictly in accordance with the New Syllabus for Higher Secondary and Multipurpose Schools for Class IX.

A TEXT BOOK OF
HIGHER SECONDARY
ELECTIVE MATHEMATICS

(IN ENGLISH)

[ALGEBRA, GEOMETRY & TRIGONOMETRY COMBINED]

PART I

(FOR CLASS IX)

INSTITUTE OF EDUCATION FOR WOMEN
2659
Dept. of Extension
SERVICE
20.3.69
CALCUTTA-27

BY

SRI KESHAB CHANDRA NAG

Retired Headmaster, Mitra Institution (Bhowanipur),
Author of S. F. & H. S. Core Math., S. F. Addl. Math.,
Patiganit (VII, VIII), Naba Patiganit (V, VI),
S. F. Aichhik Ganit (IX-X),
Core Ganit ; H. S. Elc. Math. Parts I-III,
Studies in Core Mathematics (Eng.)
Modern Arithmetic (Eng.) for Classes VII, VIII.
&
Helps to the Study of H. S. Elc. Math. Papers I-II.

NAG BOOK HOUSE
10/4C, DESHAPRAN SASMAL ROAD,
CALCUTTA-33

Published by :
Taraprasad Nag
Nag Book House
10/4C, Deshapran Sasmal Road
Calcutta-33

To be had of :
CALCUTTA BOOK HOUSE
1/1, Bankim Chatterjee Street, Calcutta-12

REVISED NEW EDITION—1968

Price Rupees Four and Fifty Paise only

Printed by : D. Sarkar, **New Basanti Press**, 17, Kailash Bose St.
Calcutta-6, Pages 1 to 160, and rest Printed by : P. C. Bhowal
Mudran Bharati (P.) Ltd. 2, Ramnath Biswas Lane
Calcutta-9

INTRODUCTION

My books "Higher Secondary Elective Mathematics" Parts I—III in Bengali for classes IX—XI have been highly spoken of by students and teachers alike. Some of my friends, teachers of experience, have often urged me to publish these popular books in English. Being thus encouraged, I have ventured to prepare English versions of the said books in three volumes under the title "A Text Book of Higher Secondary Elective Mathematics (in English)". The first part of the book (in English) has just been published and the other parts for classes X & XI will be out in due course.

This book is designed to serve as a **Text Book** for class IX of Higher Secondary Schools. It covers the entire Syllabus of Algebra, Geometry and Trigonometry for the class.

I have lucidly dealt with the topics, illustrating them by copious examples, culled from the examination papers of different Universities, to enable the students to learn them at ease.

Now the students have to buy at a time separate big volumes of Algebra, Geometry, Trigonometry, etc. at a much higher cost. I hope this book will remove that difficulty.

I shall be happy if this book is received with sympathy by the teachers and introduced as a text book in their schools.

November, 1962,
Calcutta.

Kesbab Chandra Nag,
Author.

[**Note : H. S. Examination Question Papers** and a **Chart of Important Formulas** are special features of this new edition.]

CONTENTS

Subject *Page*

ALGEBRA

1. Involution ... 1
2. Harder Factors ... 8
3. Remainder Theorem & Divisibility (Factor Theorem) 25
4. Simultaneous Equations ... 39
5. Problems on Equations ... 56
6. Theory of Indices ... 77
7. Surds ... 96
8. Evolution (Square & Cube root) ... 111
9. Quadratic Equations ... 134
10. Problems on Quadratic Equation ... 153
11. Graphs & Graphical solutions ... 164

GEOMETRY

1. Projection & Theorems on Projection ... 207
2. Ratio & Proportion ... 215
3. Problems ... 224
4. Similar Triangles ... 233

TRIGONOMETRY

1. Measurement of Angles (Different Systems) ... 260
2. Trigonometrical Ratios ... 283
3. Identities ... 287
4. Trigonometrical Ratios of some Standard Angles 298
5. Elimination ... 306
6. Solution of Equations (Simple) ... 309
7. Height and Distance ... 314
8. Answers ... 327

Appendix ... 337
Question Papers ... 363

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
CALCUTTA-27

ELECTIVE MATHEMATICS

Syllabus for Class IX

ALGEBRA :

The Remainder Theorem, Divisibility (factor theorem): Harder Factors, Laws of Indices (formal proofs for fractional and negative indices not being required); Surds; Involutions and Evolutions; Simple Simultaneous Equations and problems with two or more variables; Quadratic Equations; Graphical Solutions.

GEOMETRY :

To Prove :

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of those sides and the projection of the other side upon it.

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally and the converse.

If two triangles are equiangular, their corresponding sides are proportional and the converse.

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle and likewise the external bisector externally.

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

The ratio of the areas of similar triangles is equal to the ratio of the squares on the corresponding sides.

TRIGONOMETRY :

Measurement of angles in degrees, minutes, seconds and in radians. Definition of trigonometrical ratios of an acute angle. Trigonometrical ratios of the standard angles—0°, 30°, 45°, 60°, 90° (undefined values such as tan 90°, cot 0° to be excluded) Simple identities connecting the ratios of an angle immediately derivable from a right-angled triangle. Trigonometrical ratios of complementary angles.

Easy problems on heights and distances reducible to the solution of right-angled triangles involving the standard angles above.

SOME IMPORTANT FORMULAS & RESULTS

Algebra

1. $a^2-b^2=(a+b)(a-b)$ **2.** $a^3+b^3=(a+b)(a^2-ab+b^2)$

3. $a^3-b^3=(a-b)(a^2+ab+b^2)$

4. $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

5. $$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

6. (i) $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$

(ii) $bc(b-c)+ca(c-a)+ab(a-b)=-(a-b)(b-c)(c-a)$

(iii) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a)$

(iv) $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$$=-(a-b)(b-c)(c-a)(a+b+c)$$

(v) $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$
$$=-(a-b)(b-c)(c-a)(ab+bc+ca)$$

(vi) $$\left.\begin{array}{l} a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc, \\ bc(b+c)+ca(c+a)+ab(a+b)+3abc, \\ a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc \end{array}\right\}$$
$$\text{Each}=(a+b+c)(ab+bc+ca)$$

(vii) $$\left.\begin{array}{l} a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc, \\ bc(b+c)+ca(c+a)+ab(a+b)+2abc, \\ a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc, \\ (a+b+c)(ab+bc+ca)-abc \end{array}\right\}$$
$$\text{Each}=(a+b)(b+c)(c+a)$$

(viii) $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

7. $x-a$ is a factor of $f(x)$, if $f(a)=0$.

8. (i) a^n-b^n is exactly divisible by $a-b$, when n is a positive integer, even or odd.

(ii) a^n-b^n is exactly divisible by $a+b$, when n is an even positive integer.

(iii) a^n+b^n is exactly divisible by $a+b$, if n is an odd positive integer.

(iv) a^n+b^n is never exactly divisible by $a-b$.

9. Laws of Indices :

(i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n} (m > n)$

(v) $a^0 = 1$ (iv) $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$

(v) $(a^m)^n = a^{mn}$ (vi) $(ab)^m = a^m b^m$ (vii) $a^m b^m = (ab)^m$.

10. If $x + \sqrt{y} = a + \sqrt{b}$, where x and a are both rational, and $\sqrt{y}$ and $\sqrt{b}$ are both irrational, then $x = a$ and $y = b$.

11. If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

12. Graph : (i) The equation of a *linear graph* is $y = mx$ or $y = mx + c$.

(ii) The graph of a quadratic equation in the form $xy = 0$ is a *pair of straight lines*, $x = 0$, $y = 0$.

(iii) The equation of a *circle* is $x^2 + y^2 = a^2$ or $(x+b)^2 + (y+c)^2 = a^2$.

(iv) $y = mx^2$ or $mx = y^2$ represents a *parabola*.

(v) $ax^2 + by^2 = c^2$ represents an *ellipse*.

(vi) $ax^2 - by^2 = c^2$ represents a hyperbola and $xy = a$ or $xy + ax + by = c$ represents a rectangular hyperbola.

Trigonometry

1. $\pi = \frac{22}{7} = 3{\cdot}1416$ (App.) ; Circumference of a circle $= 2\pi r$ (r being radius), Area of a circle $= \pi r^2$.

2. 1 right-angle $= 90^\circ = 100^g = \frac{\pi^c}{2}$ (radians) ;

$1^\circ = \frac{10}{9}$ grades, $1^\circ = \frac{\pi}{180}$ radian ; $1^g = \frac{9}{10}$ degree $= \frac{\pi^c}{200}$;

$1^c = \frac{2}{\pi}$ right angle $= \frac{180^\circ}{\pi} = \frac{200^g}{\pi}$; 1 radian $= 57^\circ 17' 44{\cdot}8''$ (App.)

1 degree $= {\cdot}0174533$ radian (App.)

Circular measure of an angle $=$ *arc* $\div$ *radius*, i.e.,

$\theta = \frac{s}{r}$ where s is the arc, r the radius and θ is the angle in radian.

3. (i) $\sin\theta=\dfrac{1}{\operatorname{cosec}\theta}$, $\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$, $\therefore\ \sin\theta\times\operatorname{cosec}\theta=1$

(ii) $\cos\theta=\dfrac{1}{\sec\theta}$, $\sec\theta=\dfrac{1}{\cos\theta}$, $\therefore\ \cos\theta\times\sec\theta=1$.

(iii) $\tan\theta=\dfrac{1}{\cot\theta}$, $\cot\theta=\dfrac{1}{\tan\theta}$, $\therefore\ \tan\theta\times\cot\theta=1$.

(iv) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$.

4. (i) $\sin^2\theta+\cos^2\theta=1$, (ii) $\sec^2\theta=1+\tan^2\theta$,
(iii) $\operatorname{cosec}^2\theta=1+\cot^2\theta$.

5\.

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

6. *Complementary angles* :

$$\left.\begin{aligned}&\sin(90^\circ-\theta)=\cos\theta,\ \cos(90^\circ-\theta)=\sin\theta\ ;\\&\tan(90^\circ-\theta)=\cot\theta,\ \cot(90^\circ-\theta)=\tan\theta\ ;\\&\sec(90^\circ-\theta)=\operatorname{cosec}\theta,\ \operatorname{cosec}(90^\circ-\theta)=\sec\theta.\end{aligned}\right\}$$

7. $\sin\theta$ and $\cos\theta$ can never be numerically greater than 1 ; $\sec\theta$ and $\operatorname{cosec}\theta$ can never be numerically less than 1 ; $\tan\theta$ and $\cot\theta$ can have any value according to the value of θ.

Corrigendum

Page 5, line 15, read $\frac{15\times 4}{3}$ for $-\frac{15\times 4}{3}$;

„ 14, in sum 8, read -16 for $+16$;

„ 35, in last line of Ex. 8 read $q = -36$.

„ 48, in last line read $=3$ for $=0$.

„ 93, in sum 10 read a^2 for a^3 and

in sum 13 read b^3 for b^2.

„ 109, in sum 14(b), read $2\sqrt{5}-1$ for $2\sqrt{5}+1$.

„ 123, in 2nd line of Ex. 5, read $-2\sqrt{15}$ for $-21\sqrt{5}$.

„ 143, in Ex. 2, read 64 for 65.

„ 164, in sum 21, read 10 for 15.

„ 184, in Table, read $\frac{\pm 2}{\pm 2\cdot 6}$ for $\frac{\pm 2}{\pm 2\cdot 9}$.

„ 198, in line 5, read 55 for 50.

„ 200, in the denominator of sum 2(d) read 6 for 2.

„ 279, in sum 4(v), read π^c for π^g.

„ 284, in line 19, read P_1OM_1 for P_1OM.

„ 292, in sum 18, read $\tan^4\theta$ for $\tan^4$.

„ 293, sum 26 should be $\frac{\cot A+1}{\cot A-1} =$

„ 331, in answer 12(iv), read $\sqrt{30}$ for $\sqrt{35}$.

ALGEBRA

INVOLUTION

1. Power. The product obtained by multiplying together several factors all equal to the same number or expression is called the **power** of that number or expression.

Thus 3×3 is the **second power** of 3 and is briefly written as 3^2,
$4 \times 4 \times 4$ is the **third power** of 4 and is written as 4^3,
$a \times a \times a \times a$ is the **fourth power** of a and is written as a^4.

Again,
$$(5)^4 = 5 \times 5 \times 5 \times 5 = 625,$$
$$(x)^3 = x \times x \times x = x^3,$$
$$(a)^n = a \times a \times a \times \cdots\cdots \text{to } n \text{ factors} = a^n.$$

Index. The small figure that indicates the number of such equal factors (number or expression) is called the **Index** or **Exponent** of the power.

Thus in 3^2, 4^3, a^4 the indices are 2, 3 and 4 respectively. In $(a)^m$ or a^m, the index to the power is m.

The **second** and **third** powers of a quantity are called its **square** and **cube** respectively. The **first power** of a number is the number itself.

Thus $a^1 = a$. The index 1 is not written, we simply write a for a^1.

2. Involution. Involution is the process of finding the powers of a quantity or expression. It is the general name for multiplying an expression by itself to determine its any power.

The expression obtained by raising an expression to any power is called an expansion of the expression.

3. We know from the Rule of signs that (*i*) the even power of any expression is positive.

Thus $(+x)^2 = x \times x = x^2$; $(-x)^2 = -x \times -x = x^2$;

$$(-a)^4 = -a \times -a \times -a \times -a = a^4, \text{ etc.}$$

(*ii*) Any odd power of an expression will have the same sign as the expression itself, *i.e.*, the odd power of a positive quantity is positive and that of a negative quantity is negative.

Thus $(+3)^3 = 3 \times 3 \times 3 = 27$; $(-3a)^3 = -3a \times -3a \times -3a = -27a^3$,

$$(-x)^5 = -x \times -x \times -x \times -x \times -x = -x^5, \text{ etc.}$$

4. Rule for raising to a power :—

(*a*) To raise an expression consisting of 1 *term* only to any required power :—

(*i*) Raise the numerical coefficient, if any, to the required power by Arithmetical process and prefix the result with its proper sign, positive or negative.

(*ii*) Then raise each of the literal factors of the experession to the required power by simply multiplying its index by the index of the power.

Examples : $(2a^2)^4 = (2)^4 \cdot (a^2)^4 = 16a^8$,

$$(-2a^3b)^2 = (-2)^2 \cdot (a^3)^2 \cdot (b)^2 = 4a^6b^2,$$

$$(-3x^4y^3)^3 = (-3)^3 \cdot (x^4)^3 \cdot (y^3)^3 = -27x^{12}y^9,$$

$$\left(\frac{2ab^2}{5x^2y^2}\right)^3 = \frac{8a^3b^6}{125x^6y^6}$$ [Here the numerator and the denominator are operated upon separately]

(*b*) To raise a *Binomial* expression to any power :

By actual multiplication we have

$$\left.\begin{aligned} (a+b)^2 &= (a+b) \times (a+b) = a^2 + 2ab + b^2 \\ (a-b)^2 &= (a-b) \times (a-b) = a^2 - 2ab + b^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \end{aligned}\right\}$$

$$\left.\begin{array}{l}(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4\\(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4\end{array}\right\}$$

$$\left.\begin{array}{l}(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5\end{array}\right\}$$

Hence the Rule may be stated thus : Let a be the first term and b the second term of the given expression.

(*i*) The number of terms in the expansion of a binomial expression is one more than the index of the power.

(*ii*) By raising the first term a and the second term b to the power indicated by the index, we get the first term and the last term in the expansion. Thus in the expansion of $(a+b)^5$, the number of terms is $(5+1)$ or 6 of which a^5 is the first term and b^5 is the last term.

(*iii*) The power of a in the 2nd., 3rd. or any term in the expansion will be 1 less than its power in the just preceding term and that of b will be 1 more than its power in the preceding term.

(*iv*) To determine the numerical coefficient of any term in the expansion, multiply the numerical coefficient of the preceding term by the index of a's power in that term and then divide the product by the total number of all the preceding terms The result thus obtained is the required numerical coefficient.

(*v*) If expressions like $a+b$ and $a-b$ be raised to the same power, the only difference in the results is that all the terms in the expansion of $a+b$ will be positive, while in the expansion of $a-b$, the terms from the beginning are alternately positive and negative.

(*vi*) In any expansion the two numerical coefficients of any two terms equidistant from the beginning and the end are equal.

Examples 1

1. Expand $(a+b)^6$.

Here the index of the power is 6, so the total number of terms in the expansion will be 7, of which the first term will be a^6 and the last term b^6. The index of the first term is 6 and its numerical coefficient is 1. So to find the numerical coefficient of the 2nd term, we multiply the index 6 by the coefficient 1 of the first term and divide the result 6 by 1, as the number of terms preceding the 2nd term is only 1. Hence the second term in the expansion is $6a^5b$.

Similarly, the 3rd term $(t_3)=\frac{6\times5}{2}a^4b^2=15a^4b^2$,

the 4th term $(t_4)=\frac{15\times4}{3}a^3b^3=20a^3b^3$,

the 5th term $(t_5)=\frac{20\times3}{4}a^2b^4=15a^2b^4$,

the 6th term $(t_6)=\frac{15\times2}{5}ab^5=6ab^5$,

the 7th or last term $(t_7)=\frac{6\times1}{6}b^6=b^6$.

$$\therefore\ (a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

[*N.B.*: Here the 4th term is preceded by 3 terms and also followed by 3 terms. As the numerical coefficients of two terms equidistant from the beginning and the end are equal, the terms following the 4th term can at once be obtained without calculatian.]

2. Find the expansion of $(x-y)^7$.

Here the total number of terms in the expansion $=8$.

$$\begin{aligned}
\text{Now, }\ t_1&=x^7\\
t_2&=\tfrac{7\times1}{1}x^6(-y)=-7x^6y\\
t_3&=\tfrac{7\times6}{2}x^5(-y)^2=21x^5y^2\\
t_4&=\tfrac{21\times5}{3}x^4(-y)^3=-35x^4y^3\\
t_5&=\tfrac{35\times4}{4}x^3(-y)^4=35x^3y^4\\
t_6&=-\tfrac{35\times3}{5}x^2(-y)^5=-21x^2y^5\\
t_7&=\tfrac{21\times2}{6}xy^6=7xy^6\\
t_8&=\tfrac{7\times1}{7}(-y)^7=-y^7
\end{aligned}$$

$$\therefore\ (x-y)^7=x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7.$$

3. Determine the fifth power of $(2a+b)$.

In the expansion of $(2a+b)^5$ the total number of terms $=6$.

$$\begin{aligned} \text{Now,}\quad t_1 &= (2a)^5 = 32a^5 \\ t_2 &= 5.(2a)^4 b = 80a^4 b \\ t_3 &= \tfrac{5\times4}{2}(2a)^3 b^2 = 80a^3 b^2 \\ t_4 &= \tfrac{10\times3}{3}(2a)^2 b^3 = 40a^2 b^3 \\ t_5 &= \tfrac{10\times2}{4}(2a) b^4 = 10ab^4 \\ t_6 &= \tfrac{5\times1}{5} b^5 = b^5 \end{aligned}$$

$$\therefore\quad (2a+b)^5 = 32a^5 + 80a^4 b + 80a^3 b^2 + 40a^2 b^3 + 10ab^4 + b^5.$$

4. Expand $(1-x)^6$.

The expansion will have 7 terms in all.

$$\begin{aligned} \text{Its}\quad t_1 &= (1)^6 = 1 \\ t_2 &= 6(1)^5(-x) = -6x \\ t_3 &= \tfrac{6\times5}{2}(1)^4(-x)^2 = 15x^2 \\ t_4 &= -\tfrac{15\times4}{3}(1)^3(-x)^3 = -20x^3 \end{aligned}$$

Here the numerical coefficients of the succeeding 3 terms will be respectively equal to the numerical coefficients of t_3, t_2 and t_1.

$\therefore$ the following 3 terms are $15x^4$, $-6x^5$ and x^6.

$$\therefore\quad (1-x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6.$$

5. Expand $(x^2-2)^4$.

$$\begin{aligned} (x^2-2)^4 &= (x^2)^4 + \tfrac{4\times1}{1}(x^2)^3(-2) + \tfrac{4\times3}{2}\cdot(x^2)^2\cdot(-2)^2 \\ &\qquad + \tfrac{6\times2}{3}(x^2)(-2)^3 + \tfrac{4\times1}{4}(-2)^4 \\ &= x^8 - 8x^6 + 24x^4 - 32x^2 + 16. \end{aligned}$$

6. Find the expansion of $(2a-3b)^5$.

Here the total number of terms in the expansion $=6$.

$$\begin{aligned} \text{Now,}\quad t_1 &= (2a)^5 = 32a^5 \\ t_2 &= 5(2a)^4(-3b) = -240a^4 b \\ t_3 &= \tfrac{5\times4}{2}(2a)^3(-3b)^2 = 10(2a)^3(-3b)^2 = 720a^3 b^2 \end{aligned}$$

Hence, $t_4=10(2a)^2(-3b)^3=-1080a^2b^3$

$t_5=5(2a)(-3b)^4=810ab^4$

$t_6=(-3b)^5=-243b^5.$

$\therefore\ (2a-3b)^5=32a^5-240a^4b+720a^3b^2-1080a^2b^3$
$+810ab^4-243b^5.$

7. Simplify $(a+b)^5-(a-b)^5$.

The given expression $=(a^5+5a^4b+\frac{5\times4}{2}a^3b^2+10a^2b^3+5ab^4+b^5)$
$-(a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5)$
$=10a^4b+20a^2b^3+2b^5.$

8. If $a=3$, find the value of

$$a^6-6a^5+15a^4-20a^3+15a^2-6a-63.$$

The given exp. $=(a^6-6a^5+15a^4-20a^3+15a^2-6a+1)-64$
$=(a-1)^6-64=(3-1)^6-64$
$=(2)^6-64=64-64=0.$

[*N. B.* Here the numerical coefficients of the terms suggest that if 1 be taken as the 7th term, we get the expansion of $(a-1)^6$. Hence, -63 has been split up into $1-64$.]

9. Evaluate $a^4+8a^3+24a^2+32a+18$, when $a=\sqrt{3}-2$.

The exp. $=(a)^4+4.a^3.2+6.a^2.(2)^2+4.a.(2)^3+(2)^4+2$
$=(a+2)^4+2=(\sqrt{3}-2+2)^4+2=(\sqrt{3})^4+2$
$=(3)^2+2=9+2=11.$

10. Find the sum of the numerical coefficients of the terms in the expansion of $(x+y)^5$.

$\because\ (x+y)^5=x^5+5x^4y+\frac{5\times4}{2}x^3y^2+\frac{10\times3}{3}x^2y^3+\frac{10\times2}{4}xy^4+y^5,$

$\therefore$ the sum of the numerical coefficients
$=1+5+10+10+5+1=32.$

Otherwise: We know that any power of $1=1$. $\therefore$ if we put $x=1$, $y=1$ in the expansion of $(x+y)^5$, each term in the expansion will evidently be equal to its coefficient.

As the sum of all the terms in the expansion $=(x+y)^5$,

$\therefore$ the sum of the numerical coefficients $=(1+1)^5=(2)^5=32$.

11. Expand $(a+b+2)^4$.

$$
\begin{aligned}
(a+b+2)^4 &= \{(a+b)+2\}^4 \\
&= (a+b)^4+4(a+b)^3.2+6(a+b)^2.2^2+4(a+b).2^3+(2)^4 \\
&= (a+b)^4+8(a+b)^3+24(a+b)^2+32(a+b)+16 \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4+8(a^3+3a^2b+3ab^2+b^3) \\
&\qquad +24(a^2+2ab+b^2)+32a+32b+16 \\
&= a^4+4a^3b+8a^3+6a^2b^2+24a^2b+24a^2+4ab^3+24ab^2 \\
&\qquad +48ab+24b^2+8b^3+b^4+32a+32b+16.
\end{aligned}
$$

Exercise 1

Raise to the required power :—

1. (i) $(-2a^3b^2)^5$ (ii) $(-3x^2y^3)^6$

(iii) $\left(-\frac{ab^2}{2x^2y}\right)^5$ (iv) $(a+b)^5$

2. $(a+1)^6$ 3. $(x+y)^7$ 4. $(1+2a)^4$ 5. $(a-2)^5$

6, $(x-y)^5$ 7. $(x-1)^6$ 8. $(2a-1)^5$

Expand :—

9. $(3-c)^6$ 10. $(2a+b)^5$ 11. $(2x-3y)^5$

12. $(3a+2b)^5$ 13. $(a+1)^8$ 14. $(x-1)^9$

15. $(a^2+1)^6$ 16. $(x^2+1)^5$ 17. $(a^2-b^2)^4$

18. $(1-a^2)^6$ 19. $(x-y+z)^3$ 20. $(a+b-2)^4$

Find the sum of the numerical coefficients of the terms in the following expansions :—

21. $(a+b)^6$ 22. $(x+y)^5$ 23. $(x+b)^7$ 24. $(a+b)^8$

24. (*a*). $(2a-3b)^7$, (*b*). $(3-7x+4x^2)^3$.

Find the value of :—

25. $a^6+6a^5+15a^4+20a^3+15a^2+6a-31$, when $a=-3$.

26. $x^5+5x^4+10x^3+10x^2+5x$, when $x=2$.

27. $x^4-12x^3+54x^2-108x+81$, when $x=3$.

28. $16a^4+32a^3+24a^2+8a-80$, when $a=-2$.

29. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$, if $x=2$, $y=-1$.

30. $x^4-4x^3+6x^2-4x-2$, when $x=\sqrt{2}+1$.

31. $a^6+6a^5+15a^4+20a^3+15a^2+6a$, when $a=\sqrt[3]{3}-1$.

Simplify :—

32. $(x+y)^5-(x-y)^5$

33. $(1+x)^6-(x-1)^6$

34. $(a+b)^7+(a-b)^7$

35. $(x+a)^6-(x-a)^6$.

HARDER FACTORS

Examples 2

5. Factors of expressions in the form of a^2-b^2.

1. Factorize a^4+4b^4. [*C. U.* '22]

Here apparently the expression is not in the form a^2-b^2, but it can be reduced to that form.

Now, the given exp. $=(a^2)^2+(2b^2)^2+2.a^2.2b^2-4a^2b^2$
$=(a^2+2b^2)^2-(2ab)^2=(a^2+2b^2+2ab)(a^2+2b^2-2ab)$.

2. Factorize a^8+a^4+1.

The given exp. $=(a^4)^2+2a^4.1+(1)^2-a^4=(a^4+1)^2-(a^2)^2$

$$\begin{aligned}&=(a^4+a^2+1)(a^4-a^2+1)\\&=\{(a^2)^2+2.a^2.1+(1)^2-a^2\}\times(a^4-a^2+1)\\&=\{(a^2+1)^2-(a)^2\}(a^4-a^2+1)\\&=(a^2+a+1)(a^2-a+1)(a^4-a^2+1).\end{aligned}$$

3. Factorize $a^4-23a^2b^2+b^4$.

The given exp. $=(a^2)^2+2.a^2.b^2+(b^2)^2-25a^2b^2$

$=(a^2+b^2)^2-(5ab)^2=(a^2+5ab+b^2)(a^2-5ab+b^2)$.

4. Factorise $4a^2+b^2-c^2-d^2+4ab+2cd$. [*D. B. 1923*]

The given exp. $=(4a^2+b^2+4ab)-(c^2+d^2-2cd)$

$=(2a+b)^2-(c-d)^2=(2a+b+c-d)(2a+b-c+d)$.

5. Factorise $(a^2+b^2)(x^2-y^2)+4abxy$. [*P. U. '25, '33*]

The given exp. $=a^2x^2-a^2y^2-b^2x^2+b^2y^2+2abxy+2abxy$

$=(a^2x^2+b^2y^2+2abxy)-(a^2y^2+b^2x^2-2abxy)$

$=(ax+by)^2-(ay-bx)^2$

$=(ax+by+ay-bx)(ax+by-ay+bx)$.

6. Factorize $a^2-b^2-c^2-2bc+a-b-c$.

The given exp. $=a^2-(b^2+c^2+2bc)+a-b-c$

$=(a)^2-(b+c)^2+(a-b-c)$

$=(a+b+c)(a-b-c)+(a-b-c)=(a-b-c)(a+b+c+1)$.

7. Factorize $4x^4+1$ and hence find the two factors of 40001.

$4x^4+1=(2x^2)^2+(1)^2+2.2x^2.1-4x^2=(2x^2+1)^2-(2x)^2$

$=(2x^2+2x+1)(2x^2-2x+1)$.

Now, $40001=40000+1=4\times10000+1$

$=4\times10^4+1=4x^4+1$ [putting x for 10]

$=(2x^2+2x+1)(2x^2-2x+1)$

$=(2\times10^2+2\times10+1)\times(2\times10^2-2\times10+1)$

$=(200+20+1)(200-20+1)=221\times181$.

8. Resolve into factors $3a^2-b^2-c^2-2ab-2bc-2ca$.

The given exp. $=4a^2-a^2-b^2-c^2-2ab-2bc-2ca$

$=4a^2-(a^2+b^2+c^2+2ab+2bc+2ac)$

$=(2a)^2-(a+b+c)^2=(2a+a+b+c)(2a-a-b-c)$

$=(3a+b+c)(a-b-c)$.

9. Resolve into factors

$$2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4.$$

The exp. $=4a^2b^2-2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$

$$\begin{aligned}&=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2)\\&=(2ab)^2-(a^2+b^2-c^2)^2\\&=(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)\\&=\{(a+b)^2-(c)^2\}\{c^2-(a^2+b^2-2ab)\}\\&=(a+b+c)(a+b-c)\{(c)^2-(a-b)^2\}\\&=(a+b+c)(a+b-c)(c+a-b)(c-a+b).\end{aligned}$$

6. Factors of the expressions in the form of $a^3 \pm b^3$.

1. Resolve into factors a^6-b^6.

The given exp. $=(a^3)^2-(b^3)^2=(a^3+b^3)(a^3-b^3)$

$$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).$$

2. Factorize $8(a+b)^3-c^3$.

The given exp. $=\{2(a+b)\}^3-(c)^3=(2a+2b)^3-(c)^3$

$$\begin{aligned}&=(2a+2b-c)\{(2a+2b)^2+(2a+2b).c+c^2\}\\&=(2a+2b-c)(4a^2+4b^2+8ab+2ac+2bc+c^2).\end{aligned}$$

3. Divide $a^6+\frac{b^6}{27}$ by $a^2+ab+\frac{b^2}{3}$. [*C. U. 1930*]

$$a^6+\frac{b^6}{27}=(a^2)^3+\left(\frac{b^2}{3}\right)^3=\left(a^2+\frac{b^2}{3}\right)\left(a^4-\frac{a^2b^2}{3}+\frac{b^4}{9}\right)$$

$$\begin{aligned}&=\left(a^2+\frac{b^2}{3}\right)\left\{(a^2)^2+2a^2\cdot\frac{b^2}{3}+\left(\frac{b^2}{3}\right)^2-a^2b^2\right\}\\&=\left(a^2+\frac{b^2}{3}\right)\left\{\left(a^2+\frac{b^2}{3}\right)^2-(ab)^2\right\}\\&=\left(a^2+\frac{b^2}{3}\right)\left(a^2+ab+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right)\end{aligned}$$

$$\therefore \text{ the quotient}=\frac{\left(a^2+\frac{b^2}{3}\right)\left(a^2+ab+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right)}{\left(a^2+ab+\frac{b^2}{3}\right)}$$

$$=\left(a^2+\frac{b^2}{3}\right)\left(a^2-ab+\frac{b^2}{3}\right).$$

4. Show that $(ax+by)^3+(bx+ay)^3$ is divisible by $a+b$ and also by $x+y$. [*C. U.* '21, '26]

$$(ax+by)^3+(bx+ay)^3$$
$$=(ax+by+bx+ay)\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}$$
$$=\{a(x+y)+b(x+y)\}\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}$$
$$=(a+b)(x+y)\{(ax+by)^2-(ax+by)(bx+ay)+(bx+ay)^2\}.$$

$\because$ $a+b$ and $x+y$ are factors of the given expression,

$\therefore$ it is divisible by both $a+b$ and $x+y$.

5. Factorise $x^3-y^3+3y^2-3y+1$.

The exp. $=x^3-(y^3-3y^2+3y-1)=(x)^3-(y-1)^3$
$$=(x-y+1)\{x^2+x(y-1)+(y-1)^2\}$$
$$=(x-y+1)(x^2+xy-x+y^2-2y+1)$$
$$=(x-y+1)(x^2+y^2+1+xy-x-2y).$$

7. Factors of expressions in the form $a^3+b^3+c^3-3abc$.

1. Resolve into factors $a^3+b^3+c^3-3abc$.

The given exp. $=(a+b)^3-3ab(a+b)+c^3-3abc$
$$=(a+b)^3+(c)^3-3ab(a+b)-3abc$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c$$
$$=(a+b+c)(a^2+b^2+2ab-ac-bc+c^2-3ab)$$
$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

2. Resolve into factors $8x^3-y^3+z^3+6xyz$.

The given exp. $=(2x)^3+(-y)^3+(z)^3-3(2x).(-y).(z)$
$$=(2x-y+z)\{(2x)^2+(-y)^2+(z)^2-(2x)(-y)-(-y)(z)-(z)(2x)$$
$$=(2x-y+z)(4x^2+y^2+z^2+2xy+yz-2xz).$$

[Here factors are found with the help of the above formula. It is proper to do it as in example 1 above.]

3. Resolve into factors $x^3-y^3-1-3xy$.

The given exp, $=(x-y)^3+3xy(x-y)-1-3xy$

$$=(x-y)^3-(1)^3+3xy(x-y)-3xy$$
$$=(x-y-1)\{(x-y)^2+(x-y).1+1^2\}+3xy(x-y-1)$$
$$=(x-y-1)(x^2-2xy+y^2+x-y+1+3xy)$$
$$=(x-y-1)(x^2+y^2+1+xy+x-y).$$

4. Resolve into factors a^6+8a^3+27.

The given exp. $=a^6-a^3+27+9a^3$

$$=(a^2)^3+(-a)^3+(3)^3-3.a^2(-a).3$$
$$=(a^2-a+3)\{(a^2)^2+(-a)^2+(3)^2-(a^2)(-a)-(a^2).3-(-a).3\}$$
$$=(a^2-a+3)(a^4+a^2+9+a^3-3a^2+3a)$$
$$=(a^2-a+3)(a^4+a^3-2a^2+3a+9).$$

8. Factorization of expressions in the form of ax^2+bx+c.

Here two quantities are to be found whose product will be equal to the product of a and c (*i.e.*, the product of the coefficient of x^2 and the term independent of x), and whose sum is equal to b (*i.e.*, the coefficient of x).

1. Resolve into factors $x^2+4x-21$. [*C. U.* '16]

[Here, we have to find out two numbers whose product is 1×-21, *i.e.*, -21 and whose sum is 4. Evidently the numbers are 7 and -3. $\therefore$ we shall break up $4x$ into $7x-3x$.]

The given exp. $=x^2+7x-3x-21$

$$=x(x+7)-3(x+7)=(x+7)(x-3).$$

2. Factorize $6x^2+x-15$. [*C. U.* 1936]

[Here, we have to find out two numbers whose product is 6×-15 or -90 and sum is 1 (*i.e.*, the coefficient of x).]

The given exp. $=6x^2+10x-9x-15=2x(3x+5)-3(3x+5)$

$$=(3x+5)(2x-3).$$

3. Factorize $x^2+x-(a+1)(a+2)$.

[*N. B.* Here the product of $(a+2)$ and $-(a+1)$ is $-(a+1)(a+2)$ and their sum is $+1$. $\therefore\ x=(a+2)x-(a+1)x$]

The given exp. $=x^2+(a+2)x-(a+1)x-(a+1)(a+2)$
$=x(x+a+2)-(a+1)(x+a+2)$
$=(x+a+2)(x-a-1)$.

4. Factorize $x^2-\left(a+\frac{1}{a}\right)x+1$.

The exp. $=x^2-ax-\frac{x}{a}+1=x(x-a)-\frac{1}{a}(x-a)=(x-a)\left(x-\frac{1}{a}\right)$.

5. Factorize $(x^2-6x)^2-8(x^2-6x+8)-64$. [*B. U.* '26]

The given exp. $=a^2-8(a+8)-64$ [putting a for x^2-6x]
$=a^2-8a-64-64=a^2-8a-128$
$=a^2-16a+8a-128$
$=a(a-16)+8(a-16)=(a-16)(a+8)$
$=(x^2-6x-16)(x^2-6x+8)$ [putting the value of a]
$=(x^2-8x+2x-16)(x^2-4x-2x+8)$
$=\{x(x-8)+2(x-8)\}\{x(x-4)-2(x-4)\}$
$=(x-8)(x+2)(x-2)(x-4)$.

6. Factorize $(x+1)(x+3)(x+5)(x+7)+15$.
[*C. U.* 1941 ; *M. U.* 1926]

The exp. $=\{(x+1)(x+7)\}\{(x+3)(x+5)\}+15$
$=(x^2+8x+7)(x^2+8x+15)+15$
$=(a+7)(a+15)+15$ [putting a for x^2+8x]
$=a^2+22a+105+15$
$=a^2+22a+120=a^2+12a+10a+120$
$=a(a+12)+10(a+12)=(a+10)(a+12)$
$=(x^2+8x+10)(x^2+8x+12)$ [putting the value of a]
$=(x^2+8x+10)(x^2+6x+2x+12)$
$=(x^2+8x+10)\{x(x+6)+2(x+6)\}$
$=(x^2+8x+10)(x+6)(x+2)$.

[*N. B.* Here in the products of $(x+1)(x+7)$ and of $(x+3)(x+5)$, the terms containing x^2 and x are the same. So they are separately grouped together.]

7. Factorize $(x+1)(x+3)(x-4)(x-6)+24$. [*D. B.* '22]

The given exp. $=\{(x+1)(x-4)\}\{(x+3)(x-6)\}+24$

$=(x^2-3x-4)(x^2-3x-18)+24$

$=(a-4)(a-18)+24$ [putting a for x^2-3x]

$=a^2-22a+96=a^2-16a-6a+96=a(a-16)-6(a-16)$

$=(a-6)(a-16)=(x^2-3x-6)(x^2-3x-16)$.

9. Factorization by the trial method.

1. Factorize x^3-3x+2. [*C. U.* 1930 ; *C. B.* 1916]

By trial find that value of x, for which the given expression is zero. Here evidently the expression is 0, if $x=+1$.

$\therefore$ $x-1$ will be a factor of the given expression.

Now, $x^3-3x+2=x^3-x^2+x^2-x-2x+2$

$=x^2(x-1)+x(x-1)-2(x-1)$

$=(x-1)(x^2+x-2)=(x-1)(x^2+2x-x-2)$

$=(x-1)\{x(x+2)-1(x+2)\}$

$=(x-1)(x+2)(x-1)=(x-1)^2(x+2)$.

2. Factorize x^3-x^2-7x-2.

Here, if $x=-2$, the given exp. $=0$. Hence $x+2$ is a factor of the expression.

Now, the given exp. $=x^3+2x^2-3x^2-6x-x-2$

$=x^2(x+2)-3x(x+2)-1(x+2)=(x+2)(x^2-3x-1)$.

3 Factorize $x^3-6x^2+11x+6$. [*A. U.* 1921]

Here if $x=1$, the given exp. $=0$.

$\therefore$ $x-1$ is a factor of the expression.

$\therefore$ $x^3-6x^2+11x-6=x^3-x^2-5x^2+5x+6x-6$

$=x^2(x-1)-5x(x-1)+6(x-1)$

$=(x-1)(x^2-5x+6)=(x-1)(x^2-3x-2x+6)$

$=(x-1)\{x(x-3)-2(x-3)\}=(x-1)(x-3)(x-2)$.

4. Factorize $8a^3+4a-3$.

The exp. $=8a^3-1+4a-2=\{(2a)^3-(1)^3\}+2(2a-1)$

$$=(2a-1)(4a^2+2a+1)+2(2a-1)$$
$$=(2a-1)(4a^2+2a+1+2)=(2a-1)(4a^2+2a+3).$$

5. Factorize x^5+y^5.

If $-y$ be put for x, the given exp. $=0$, so $x+y$ is a factor of the given expression.

Now, $x^5+y^5=x^5+x^4y-x^4y-x^3y^2+x^3y^2+x^2y^3-x^2y^3-xy^4+xy^4+y^5$

$$=x^4(x+y)-x^3y(x+y)+x^2y^2(x+y)-xy^3(x+y)+y^4(x+y)$$
$$=(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4).$$

6. Factorize $x^4-x^3-7x^2+x+6$.

Here we find that if x have any of the values $1, -1, -2, 3$, the given expression $=0$; $\therefore$ $x-1, x+1, x+2, x-3$ are factors of the expression.

$\therefore$ The terms are so arranged that $x+1$ may be a factor of the expression.

Now, the exp. $=x^4+x^3-2x^3-2x^2-5x^2-5x+6x+6$

$$=x^3(x+1)-2x^2(x+1)-5x(x+1)+6(x+1)$$
$$=(x+1)(x^3-2x^2-5x+6)$$

[Again arrange the 2nd factor so that $(x-1)$ is a factor of it]

$$=(x+1)(x^3-x^2-x^2+x-6x+6)$$
$$=(x+1)\{x^2(x-1)-x(x-1)-6(x-1)\}$$
$$=(x+1)(x-1)(x^2-x-6)$$

[Arrange the 3rd factor so that $x+2$ is one of its factors]

$$=(x+1)(x-1)(x^2+2x-3x-6)$$
$$=(x+1)(x-1)\{x(x+2)-3(x+2)\}$$
$$=(x+1)(x-1)(x+2)(x-3).$$

10. Factorization by arranging the terms in the descending powers of the letters involved.

1. Factorize $a^2-2b^2-3c^2+ab+2ac+7bc$.

Here first arrange the terms in the descending powers of any of the letters a, b, c.

$$\begin{aligned}\text{Thus, the given exp.}&=a^2+ab+2ac-2b^2+7bc-3c^2\\&=a^2+(b+2c)a-(2b^2-7bc+3c^2)\\&=a^2+(b+2c)a-(2b^2-6bc-bc+3c^2)\\&=a^2+(b+2c)a-(2b-c)(b-3c)\\&=a^2+(2b-c)a-(b-3c)a-(2b-c)(b-3c)\\&=a(a+2b-c)-(b-3c)(a+2b-c)\\&=(a+2b-c)(a-b+3c).\end{aligned}$$

[*N. B.* Here $(b+2c)$, *i.e.*, the coefficient of a has been split up into two expressions $(2b-c)$ and $-(b-3c)$, so that their sum is $b+2c$ and product is $-(2b-c)(b-3c)$.]

2. Factorize $a^2-3bc+3b^2-4ab+ac$.

$$\begin{aligned}\text{The given exp.}&=(a^2-4ab+3b^2)+ac-3bc\\&=(a^2-ab-3ab+3b^2)+c(a-3b)\\&=(a-b)(a-3b)+c(a-3b)=(a-3b)(a-b+c).\end{aligned}$$

[*N. B.* Here the terms in which a, b are of the same degree have been grouped together. The remaining terms are grouped together.]

3. Resolve into factors $a^4+a^3-10a^2+a+1$.

[*N. B.* To factorize an expression in which the coefficients of the terms equidistant from the beginning and the end are equal, such terms are grouped together. Such expressions are called *reciprocal expressions.*]

Here the given exp. $=(a^4+1)+(a^3+a)-10a^2$

$=(a^2+1)^2-2a^2+a(a^2+1)-10a^2=(a\;+1)^2+a(a^2+1)-12a^2$

$=x^2+ax-12a^2$ [putting x for a^2+1]

$=x^2+4ax-3ax-12a^2=\{x(x+4a)-3a(x+4a)\}$

$=(x+4a)(x-3a)=(a^2+4a+1)(a^2-3a+1)$.

[Factors of cyclic order]

11. Cyclic order : If the letters a, b and c are arranged round a circle and they are taken (beginning with anyone of them) in the clockwise direction round the circumference, the order in which the letters occur is called the *cyclic order*. Thus abc ; bca : cab ; $a-b$, $b-c$, $c-a$; or $b-c$, $c-a$, $a-b$; or $c-a$, $a-b$, $b-c$ are in cyclic order.

The letters may be x, y, z instead of a, b, c.

To factorize an expression where the terms are in cyclic order, we have to arrange the terms first in the descending power of any of the letters involved (say a or x), When the first factor is obtained, the second may be arranged in the descending powers of another letter (b or y) and so on.

4. Factorize $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

[*C. U.* 1928, '40, '45 ; *D B.* '24 ; *G. U.* '48]

The given exp. $=a^2(b-c)+b^2c-ab^2+ac^2-bc^2$

$=a^2(b-c)-a(b^2-c^2)+bc(b-c)=(b-c)\{a^2-a(b+c)+bc\}$

$=(b-c)(a^2-ab-ac+bc)=(b-c)\{a(a-b)-c(a-b)\}$

$=(b-c)(a-b)(a-c)=-(a-b)(b-c)(c-a)$.

[Here of the factors $(b-c)(a-b)(a-c)$ all are not in cyclic order. The convention is to give the factors in the cyclie order Here the factor $a-c$ is not in that order, so it is written as $-c+a$ or $-(c-a)$. Hence the factors are $-(a-b)(b-c)(c-a)$,

If a, b, c be replaced by a^2, b^2, c^2 respectively in the above sum, we get the facrors of the expression

$a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.]

5. Factorize $bc(b-c)+ca(c-a)+ab(a-b)$. [*P. U.* '30]

The given exp. $=bc(b-c)+c^2a-ca^2+a^2b-ab^2$

$=bc(b-c)+a^2b-a^2c-ab^2+ac^2$

$=bc(b-c)+a^2(b-c)-a(b^2-c^2)=(b-c)(bc+a^2-ab-ac)$

$=(b-c)\{a(a-c)-b(a-c)\}=(b-c)(a-b)(a-c)$

$=-(a-b)(b-c)(c-a)$.

6. Factorize $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

[*D. B.* 1930]

The exp. $=a(b^2-c^2)+bc^2-a^2b+a^2c-b^2c$

$=a(b^2-c^2)-a^2(b--c)-bc(b-c)$

$=(b-c)(ab+ac-a^2-bc)=(b-c)\{a(c-a)-b(c-a)\}$

$=(b-c)(c-a)(a-b)=(a-b)(b-c)(c-a)$.

7. Factorize $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

The given exp. $=a^3(b-c)+b^3c-ab^3+ac^3-bc^3$

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$ [arranged in descending powers of *a*]

$=a^3(b-c)-a(b-c)(b^2+bc+c^2)+bc(b+c)(b-c)$

$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$

$=(b-c)(a^3-ab^2-abc-ac^2+b^2c+bc^2)$

$=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$ [arranged in descendidg powers of *b*]

$=(b-c)(c-a)(b^2+bc-ac-a^2)$

$=(b-c)(c-a)\{c(b-a)+(b^2-a^2)\}$

$=(b-c)(c-a)(b-a)(c+b+a)$

$=-(a-b)(b-c)(c-a)(a+b+c)$.

8. Factorize $(b-c)^3+(c-a)^3+(a-b)^3$. [*C. U.* '29. '39]

Let $x=b-c,\ y=c-a,\ z=a-b$.

Then $x+y+z=b-c+c-a+a-b=0$.

Now, $(b-c)^3+(c-a)^3+(a-b)^3$

$=x^3+y^3+z^3=x^3+y^3+z^3-3xyz+3xyz$

$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$=0\times(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$=0+3xyz=3xyz=3(b-c)(c-a\ (a-b)$.

9. Factorize $(x+a)^2(b-c)+(x+b)^2(c-a)+(x+c)^2(a-b)$.

The given exp. $=(x^2+2ax+a^2)(b-c)+(x^2+2bx+b^2)(c-a)$
$\qquad +(x^2+2cx+c^2)(a-b)$
$=x^2(b-c+c-a+a-b)+2x\{a(b-c)+b(c-a)+c(a-b)\}$
$\qquad +a^2(b-c)+b^2(c-a)+c^2(a-b)$
$=x^2\times 0+2x\times 0+a^2(b-c)+b^2(c-a)+c^2(a-b)$
$=a^2(b-c)+b^2(c-a)+c^2(a-b)$
$=-(a-b)(b-c)(c-a)$.

10. Resolve into factors

$a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc.$ [*C. U.* '32, '35]

The exp. $=(a^2b+a^2c+abc)+(b^2c+ab^2+abc)+(c^2a+c^2b+abc)$
$=a(ab+ac+bc)+b(bc+ab+ac)+c(ac+bc+ab)$
$=(ab+ac+bc)(a+b+c)$.

[Out of $3abc$ one abc is taken with each term]

11. Factorize $bc(b+c)+ca(c+a)+ab(a+b)+3abc$.

[*C. U.* 1939]

The given exp. $=bc(b+c)+abc+ca(c+a)+abc+ab(a+b)+abc$
$=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$
$=(a+b+c)(bc+ca+ab)$.

12. Factorize $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$.

The given exp. $=(a^2b+a^2c+abc)+(ab^2+abc+b^2c)$
$\qquad +(abc+ac^2+bc^2)$
$=(a+b+c)(ab+bc+ca)$, [see sum 10]

13. Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

[*C. U.* '33, '50]

The exp. $=a^2(b+c)+b^2c+ab^2+ac^2+bc^2+2abc$
$=a^2(b+c)+(ab^2+ac^2+2abc)+(b^2c+bc^2)$
$=a^2(b+c)+a(b+c)^2+bc(b+c)=(b+c)(a^2+ab+ac+bc)$
$=(b+c)\{a(a+b)+c(a+b)\}=(b+c)(c+a)(a+b)$.

14. Factorize $bc(b+c)+ca(c+a)+ab(a+b)+2abc$.

The exp. $=bc(b+c)+ac^2+a^2c+a^2b+ab^2+2abc$

$=bc(b+c)+(a^2b+a^2c)+(ab^2+ac^2+2abc)$

$=bc(b+c)+a^2(b+c)+a(b+c)^2=(b+c)(bc+a^2+ab+ac)$

$=(b+c)\{a(a+b)+c(a+b)\}=(b+c)(c+a)(a+b)$.

15. Factorize $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc$.

The exp. $=a(b^2+c^2)+2abc+a^2b+a^2c+b^2c+bc^2$

$=a(b^2+c^2+2bc)+a^2(b+c)+bc(b+c)$

$=a(b+c)^2+a^2(b+c)+bc(b+c)$

$=(b+c)(ab+ac+a^2+bc)=(b+c)(c+a)(a+b)$.

16. Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3+c^3$.

The exp. $=\{a^2(b+c)+a^3\}+\{b^2(c+a)+b^3\}+\{c^2(a+b)+c^3\}$

$=a^2(b+c+a)+b^2(c+a+b)+c^2(a+b+c)$

$=(a+b+c)(a^2+b^2+c^2)$.

17. Factorize $(x+y+z)^3-x^3-y^3-z^3$.

The exp. $=x^3+y^3+z^3+3(x+y)(y+z)(z+x)-x^3-y^3-z^3$

$=3(x+y)(y+z)(z+x)$.

18. Factorize $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc$.

[*A. U.* 1922]

The given exp. $=a(b^2-2bc+c^2)+b(c^2-2ca+a^2)$

$+c(a^2-2ab+b^2)+9abc$

$=b^2(c+a)+c^2(a+b)+a^2(b+c)-2abc-2abc-2abc+9abc$

$=b^2(c+a)+c^2(a+b)+a^2(b+c)+3abc$

$=(a+b+c)(ab+bc+ca)$. [see sum 10]

19. Factorize $(a+b+c)(ab+bc+ca)-abc$.

The given exp. $=\{a+(b+c)\}\{a(b+c)+bc\}-abc$

$=a^2(b+c)+a(b+c)^2+bc(b+c)+abc-abc$

$=(b+c)(a^2+ab+ac+bc)=(b+c)(c+a)(a+b)$.

20. Factorize $(b+c)(c+a)(a+b)+abc$.

The given exp. $=a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+2abc+abc$

$=(a^2b+ab^2+abc)+(b^2c+bc^2+abc)+(a^2c+ac^2+abc)$

$=ab(a+b+c)+bc(b+c+a)+ca(a+c+b)$

$=(a+b+c)(ab+bc+ca)$.

21. Factorize $(3x-2y)^3-(2x-3y)^3-(x+y)^3$.

Let $a=2x-3y$ and $b=x+y$; then $a+b=3x-2y$.

$\therefore$ The given exp. $=(a+b)^3-a^3-b^3=a^3+b^3+3ab(a+b)-a^3-b^3$

$=3ab(a+b)=3(2x-3y)(x+y)(3x-2y)$.

22. Factorize $x^3+(x-1)^3+(1-2x)^3$. [*D. B.* 1940]

Suppose $a=x$, $b=x-1$, $c=1-2x$.

Then, $a+b+c=x+x-1+1-2x=0$.

Now, the given exp. $=a^3+b^3+c^3=3abc$ [$\because a+b+c=0$]

$=3x(x-1)(1-2x)$ [putting the values of a, b, c]

23. Factorize $a(b-c)x^2+b(c-a)x+c(a-b)$.

[*D. B.* 1933]

The given exp. $=(ab-ac)x^2+(bc-ab)x+(ac-bc)$

$=(ab-ac)x^2-(ab-ac)x-(ac-bc)x+(ac-bc)$

[$\because\ -(ab-ac)-(ac-bc)=(bc-ab)$]

$=(ab-ac)x(x-1)-(ac-bc)(x-1)$

$=(x-1)\{(ab-ac)x-(ac-bc)\}=(x-1)(abx-acx-ac+bc)$.

24. Factorize $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$.

The given exp. $=(2a+2b+2c)^3-(a+b)^3-(b+c)^3-(c+a)^3$

$=\{(a+b)+(b+c)+(c+a)\}^3-(a+b)^3-(b+c)^3-(c+a)^3$

$=(a+b)^3+(b+c)^3+(c+a)^3+3(a+b+b+c)$

$\times(b+c+c+a)(c+a+a+b)-(a+b)^3-(b+c)^3-(c+a)^3$

$=3(a+2b+c)(a+b+2c)(2a+b+c)$.

25. Factorize $(2b-a)^3+(2a-b)^3-(a+b)^3$. [*B. U.* 1900]

Let $x=2b-a$ and $y=2a-b$, $\therefore\ x+y=a+b$.

Now, the given exp. $=x^3+y^3-(x+y)^3$

$$=x^3+y^3-\{x^3+y^3+3xy(x+y)\}$$
$$=x^3+y^3-x^3-y^3-3xy(x+y)=-3xy(x+y)$$
$$=-3(2b-a)(2a-b)(a+b)$$
$$=3(a-2b)(2a-b)(a+b) \qquad [\because\ -(2b-a)=(a-2b)]$$

26. Factorize $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$. [*I. P. S.* 1936]

Let $x=a+b-2c$, $y=b+c-2a$, $z=c+a-2b$.

Then $x+y+z=a+b-2c+b+c-2a+c+a-2b=0$.

Now, the given exp. $=x^3+y^3+z^3=3xyz \quad [\because\ x+y+z=0]$

$$=3(a+b-2c)(b+c-2a)(c+a-2b).$$

27. Factorize $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.

The given exp. $=(ab-ac)^3+(bc-ab)^3+(ac-bc)^3$

$=3(ab-ac)(bc-ab)(ac-bc)$ $[\because\ ab-ac+bc-ab+ac-bc=0]$

$=3abc(b-c)(c-a)(a-b)$.

28. Factorize $2a^2+2a-3ab-b+b^2$. [*D. B.* 1949]

The given exp. $=(2a^2-3ab+b^2)+2a-b$

$$=(2a^2-2ab-ab+b^2)+1(2a-b)$$
$$=2a(a-b)-b(a-b)+1(2a-b)$$
$$=(2a-b)(a-b)+1(2a-b)=(2a-b)(a-b+1).$$

Exercise 2

Resolve into factors :—

[*Group A*]

1. $m^4+m^2n^2+n^4$ [*C. U.* '23] 2. x^4+4 [*C. U.* '34]

3. x^4+64 [*S. F.* '52] 4. $4x^4+81$ [*C. U.* '37]

5. x^4+2x^2+9 [*D. B.* '30]

6. x^4-8x^2+4 [*B. U.* '24]

7. $4x^2-4xy-2yz-z^2$ [*C. U.* '35]

8. $a^2-b^2-c^2+d^2-2(ad-bc)$ [*S. F.* '53]

9. $4b^2c^2-(b^2+c^2-a^2)^2$ [*B. U.* '23]

10. $x^2-y^2-6xa+2ya+8a^2$ [*D. B.* '48]

11. $2(ab+cd)-a^2-b^2+c^2+d^2$.

12. $4(ad-bc)^2-(a^2+d^2-b^2-c^2)^2$.

13. $a^4b^4+1+a^2b^2-c^2+2abc$.

[*Group B*]

14. x^6-729 [*D. B.* '37]

15. $x^3-y^3+6y^2-12y+8$.

16. $2a^3+3a^2+3a+1$.

17. $(a+2b-c)^3-(a+b)^3-(b-c)^3$

[Hints : $a+2b-c=(a+b)+(b-c)$]

[*Group C*]

18. $x^3-y^3+z^3+3xyz$ [*A. U.* '24]

19. $a^3+b^3-8c^3+6abc$

20. $8a^3+b^3-27+18ab$.

21. a^6+32a^3-64.

22. x^6-22x^3+125.

23. a^6+5a^3+8.

24. $(2x-3y)^3+(3y-z)^3+(z-2x)^3$.

25. $(a+b+c)^3-a^3-b^3-c^3$.

[*Group D*]

26. $12x^2+65x+77$.

27. $21x^2+40xy-21y^2$.

28. $8a^4+2a^2-45$ [*A. U.* '27]

29. $x^2+2x-(a+1)(a+3)$.

30. $(a+b)^2-10(a^2-b^2)-56(a-b)^2$ [*B. U.* '34]

31. $(x^2-4x)(x^2-4x-1)-20$ [*B. U.* '12]

[*Group E*]

32. $a^3-3a-2a^2+4$ [*C. U.* 39]

33. $x^3+16x^2+13x-30$ [*D. B.* '36]

34. $x^3+9x^2+26x+24$ [*D. B.* '38]

35. $(ax+by)^2+(bx-ay)^2$. [*P. U.* '34]

36. $(1-c^2)(1+a)^2-(1-a^2)(1-c)^2$

37. $x(x-1)(x-2)-3x+3$ [*D. B.* '36]

38. $a(a-1)(a-2)(a-3)-120$.

39. $(x+1)(x+3)(x-4)(x-6)+13$.

40. $x(2x+1)(x-2)(2x-3)-63$.

41. $(x+1)(x+2)(x+3)(x+4)-3$. [*C. U.* '46]

42. $(x+y+z)(xy+yz+zx)-xyz$. [*D. B.* '32]

43. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$. [*M. U.* '27]

44. $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$.

45. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.

46. $4(x^2+2x+5)^2+17(x^2+2x+5)(x^2+6x)+4(x^2+6x)^2$.

47. $a^4+6a^3+4a^2-15a+6$ [*C. U.* '48]

48. $x^4+x^3+2x^2+x+1$ [*C. U.* '42]

49. $x^2(b-c)+b^2(c-x)+c^2(x-b)$. [*D. B.* '27]

50. $6a^2-2b^2-5bc-ab+11ac+3c^2$.

51. $a^4-5a^3b+6a^2b^2-5ab^3+b^4$.

52. $x^5+4x^4-13x^3-13x^2+4x+1$.

53. $xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)$.

54. x^7+y^7.

55. $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$

56. $(a-b)^5+(b-c)^5+(c-a)^5$.

THE REMAINDER THEOREM AND DIVISIBILITY

12. Functions : Any expression involving a letter is called a *function* of that letter.

The letters involved in any function are called its **variables.**

The value of the function depends on those of the letters involved.

Thus a, $2a+5$, $2a^2+3a+1$ are functions of a, where a is the variable. Similarly x, $2x+3$, x^2-2x+3 are functions of x (x being the variable), and $2a^2+b^2-3ab$ is a function of a and b. In the last example a and b are the variables.

The function of x is symbolically denoted by $f(x)$ or $F(x)$.

As $2x^2+3x-4$ is a function, we have $f(x)=2x^2+3x-4$.

Now if any particular value be given to x (say 2), we have

$f(2)=2.2^2+3.2-4$. Similary $f(0)=2.0^2+3.0-4$.

Rational and Integral function : A function is called *rational* when it has no term containing a square root or any other root. When the powers of the letter or letters in a function are all positive integers, it is called an *integral* function.

Thus $2x^2+3x-4$ and $3x^3-2x^2+5x+4$ are rational and integeral functions of x.

If in the expression $px^n+qx^{n-1}+rx^{n-2}+\ldots\ldots+lx+m$, n is a positive integer and each of $p, q, r, \cdots\cdots, l, m$ is a constant, then it is a rational and integral function of x.

13. Remainder Theorem :

The remainder (independent of x) in the division of any rational and integral function of x by $x-a$, is obtained by putting a for x in the function.

Proof : (First method) : Let mx^2+nx+l be the function. By actual division we have

$$\begin{array}{l|l|l} x-a & mx^2+nx+l & mx+(ma+n) \\ & \underline{mx^2-max} & \\ & (ma+n)x+l & \\ & \underline{(ma+n)x-a(ma+n)} & \\ & ma^2+na+l \text{ (which is the remainder independent of } x) & \end{array}$$

If we put a for x in the function mx^2+nx+l we have ma^2+na+l, which is the remainder.

Thus the theorem is proved.

(Second method) : Let Q be the quotient and R the remainder (independent of x) when mx^2+nx+l is divided by $x-a$.

We have to prove that $R=ma^2+na+l$.

Proof : $\because$ Dividend = divisor × quotient + remainder,

$\therefore$ here $mx^2+nx+l=(x-a)\times Q+R$, and as it is an identity, it will be satisfied by any value of x, and R being independent of x will remain constant.

Now suppose the value of Q becomes Q' when $x=a$.

$\therefore$ $ma^2+na+l=(a-a)\times Q'+R=0\times Q'+R=R$.

Hence the theorem is proved.

[*N.B.* (1) The remainder in the division of a function of x by $x-a$, is obtained by putting a for x in the function.

(2) In the division of a function by $x+a$, the remainder is obtained by putting $-a$ for x ;

(3) In the division of a function by $bx+a$, the remainder is found by putting $-\frac{a}{b}$ for x.]

Examples. Find without actual division, the remainders in the following :

(a). $(x^3+2x^2-3x+5)\div(x-3)$

(b). $(px^2+qx+r)\div(x+a)$.

(a) Here the remainder will be found by putting 3 for x in the dividend.

$\therefore$ the required remainder $=(3)^3+2.(3)^2-3.3+5$

$=27+18-9+5=41.$

(b) $\because$ The divisor is $x+a$, $\therefore$ the remainder is obtained by putting $-a$ for x in the dividend.

$\therefore$ the reqd. remainder $=p(-a)^2+q(-a)+r=pa^2-qa+r.$

THE FACTOR THEOREM

If any rational integral function of x becomes equal to 0 when a is substituted for x, it contains $x-a$ as a factor. It is therefore exactly divisible by $x-a$.

[In symbols : $x-a$ is a factor of $f(x)$, if $f(a)=0$]

$\because$ the expression becomes zero, when a is written for x,

$\therefore$ by the remainder theorem we have that the remainder on division $=f(a)=0$.

$\therefore$ The expression is exactly divisible by $x-a$ and hence $x-a$ is a factor of the expression.

[*N. B.* (1) If an expression becomes zero, when $-a$ is substituted for x, then it has $x-(-a)$ or $x+a$ as a factor and is divisible by $x+a$.

(2) If an expression becomes 0, when $x=-\frac{b}{a}$, it has $ax+b$ as a factor.]

Examples 3

Ex 1. Prove that $x^3-7x^2+11x-2$ is exactly divisible by $x-2$.

Here the divisor is $x-2$. $\therefore$ The given expression will be divisible by $x-2$, if it becomes 0, when $x=2$.

Now, $x^3-7x^2+11x-2=(2)^3-7(2)^2+11\times2-2$

$$=8-28+22-2=0,$$

$\therefore$ The given expression is divisible by $x-2$.

Ex. 2. Show that $x-1$ is a factor of x^4-3x+2.

By putting $x=1$ in x^4-3x+2, we have

$$x^4-3x+2=(1)^4-3\times1+2=1-3+2=0,$$

$\therefore$ $x-1$ is a factor of the given expression.

Ex. 3. Find for what value of p the expression $x^5-61x+p$ is exactly divisible by $x+1$.

Here the divisor is $x+1$, so the given expression will be divisible by $x+1$, if it becomes 0, when $x=-1$.

Now, $x^5-61x+p=(-1)^5-61\times(-1)+p=-1+61+p$

$$=60+p.$$

$\therefore$ $60+p$ must be equal to 0, $\therefore$ $p=-60$ which is the required value of p.

Ex. 4. Find the relation between b and c so that x^3+bx+c and x^3+cx+b may have a common factor. [*P. U.* 1891]

Let $x-a$ be the common factor.

Now, $\because$ $x-a$ is a factor of x^3+bx+c,

$\therefore$ $a^3+ab+c=0 \cdots\cdots (1)$.

Again, $\because$ $x-a$ is a factor of x^3+cx+b,

$\therefore$ $a^3+ac+b=0 \cdots\cdots (2)$.

Subtracting (2) from (1) we have $ab-ac-b+c=0$,

or, $a(b-c)=b-c$, $\therefore$ $a=\dfrac{b-c}{b-c}=1$.

Now, putting $a=1$ in (1) we have $1^3+1.b+c=0$

or, $b+c+1=0$, which is the relation between b and c.

Ex. 5. What value (not zero) must a have, if x^2+x-a and x^3-x-a have a common factor? [*P. U.* '25]

Let $x-p$ be the common factor.

$\because$ $x-p$ is a factor of both the given expressions,

$\therefore$ we have $p^2+p-a=0$......(1) and $p^3-p-a=0$......(2)

From (1)−(2) we have $-p^3+p^2+2p=0$,

or, $-p(p^2-p-2)=0$, or, $-p(p-2)(p+1)=0$,

$\therefore$ p may be equal to 0, −1, or 2.

Now, by putting the value of p in (1) we shall have the value of a.

By putting $p=0$, or $p=1$ in (1) we have $a=0$, which is inadmissible according to the given condition.

Now, putting $p=2$ in (1) we get $2^2+2-a=0$, $\therefore$ $a=6$.

$\therefore$ The required value of a is 6.

Ex. 6. Find the condition that $x^3+(p+q)x+a$ may be divisible by $x+p+q$. [*P. U.* 1895]

The given expression will be divisible by $x+p+q$, if it becomes zero when $x+p+q=0$, *i.e.*, when $x=-(p+q)$.

Now, $\{-(p+q)\}^3+(p+q)\times-(p+q)+a=0$,

or, $-(p+q)^3-(p+q)^2+a=0$, or, $(p+q)^3+(p+q)^2-a=0$,

or, $(p+q)^3+(p+q)^2=a$, or, $(p+q)^2(p+q+1)=a$.

$\therefore$ The required condition is $a=(p+q)^2(p+q+1)$.

Ex. 7. If $x+a$ be the H. C. F. of px^2+qx+r and rx^2+qx+p, prove that either $p+q+r=0$, or, $q=p+r$.

$\because$ $x+a$ is the H. C. F. of the two given expressions,

$\therefore$ they will become 0, if $x+a=0$ or if $x=-a$.

Putting this value of a in the expressions,

we have $pa^2-qa+r=0 \quad \cdots(1)$ and $ra^2-qa+p=0 \quad \cdots(2)$

From (1)−(2), we have $pa^2-ra^2-p+r=0$,

or, $a^2(p-r)-(p-r)=0$, or, $a^2(p-r)=p-r$, or, $a^2=1$,

$\therefore$ $a=\pm 1$.

Now, if $a=1$, then from (1) we have $p(1)^2-q(1)+r=0$,

or, $p-q+r=0$. $\therefore$ $q=p+r$.

Again, if $a=-1$, from (1) we have $p(-1)^2-q(-1)+r=0$,

or, $p+q+r=0$.

Hence, if $x+a$ be the H. C. F. of the given expressions, then either $p+q+r=0$ or $q=p+r$.

Ex. 8. Show that $(ax+by)(bx+cy)(cx+ay)-(ay+bx)\times(by+cx)(cy+ax)$ contains the factors $(x-y)$, $(a-b)$, $(b-c)$ and $(c-a)$.

[M.U.]

$(x-y)$ will be a factor of the given expression, if the expression becomes zero, when the value of x is y.

Now putting x for y, the given exp. $=(ax+bx)(bx+cx)(cx+ax)-(ax+bx)(bx+cx)(cx+ax)=0$.

$\therefore$ $x-y$ is a factor of the expression.

Similarly it can be proved that $a-b$, $b-c$, $c-a$ are each a factor of the given expression.

15. Some Theorems on Divisibility.

Theorem 1. When n is a positive integer, even or odd, the expression a^n-b^n is always exactly divisible by $a-b$.

Proof : Suppose Q to be the quotient and R the remainder, which does not contain a, when a^n-b^n is divided by $a-b$.

We have $a^n-b^n=(a-b)\times Q+R$ (identically).

$\because$ R does not contain a, $\therefore$ it will remain the same whatever value a may have. But as Q is not free from a, the value of Q will change with the value of a.

Suppose the value of Q is Q', when $a=b$.

Now putting b for a in the above identity, we have

$$b^n-b^n=(b-b)\times Q'+R=0\times Q'+R=R.$$

$\therefore$ $R=b^n-b^n=0$. Hence a^n-b^n is divisible by $a-b$.

[*N. B.* (*i*) $\dfrac{a^n-b^n}{a-b}=\dfrac{a-b}{a-b}=1$ (when $n=1$);

(*ii*) $\dfrac{a^n-b^n}{a-b}=\dfrac{a^2-b^2}{a-b}=a+b$ (when $n=2$);

(*eii*) $\dfrac{a^n-b^n}{a-b}=\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$ (when $n=3$);

(*iv*) generally, $\dfrac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}.b+a^{n-3}.b^2+\cdots+b^{n-1}$.]

Theorem 2. The expression a^n-b^n is exactly divisible by $a+b$, when n is an even positive integer, but not when n is any odd positive integer.

(As in the proof of Theorem 1 above, we have)

$$a^n-b^n=(a+b)\times Q+R \text{ (identically).}$$

$\because$ R does not contain a,

$\therefore$ R remains the same whatever may be the value of a.

Putting $-b$ for a in the above identity, we have

$$(-b)^n-b^n=(-b+b)\times Q'+R=0\times Q'+R=R.$$

Now, if n is an even integer, $(-b)^n=b^n$ and in that case $(-b)^n-(b)^n=b^n-b^n=0$, and then R becomes zero.

Hence a^n-b^n is exactly divisible by $a+b$, when n is an even positive integer.

Again, if n be an odd positive integer, $(-b)^n=-b^n$, and in that case $(-b)^n-b^n=-b^n-b^n=-2b^n$, and so R does not become 0.

Hence a^n-b^n is not exactly divisible by $a+b$, when n is an odd integer.

Theorem 3. The expression a^n+b^n is exactly divisible by $a+b$, if n be an odd positive integer, but not if n be even.

As before, we have $a^n+b^n=(a+b)\times Q+R$ (identically).

Putting $-b$ for a in the identity, we have

$$(-b)^n+b^n=(-b+b)\times Q'+R=0\times Q'+R=R.$$

Now, if n be an odd integer we have $(-b)^n=-b^n$

$$\text{and then } R=(-b)^n+b^n=-b^n+b^n=0,$$

$\therefore$ a^n+b^n is exactly divisible by $a+b$, when n is an odd positive integer.

Again, if n be even, $(-b)^n=b^n$, and then

$$R=(-b)^n+b^n=b^n+b^n=2b^n, \text{ but } R \text{ is not } 0.$$

$\therefore$ a^n+b^n is not exactly divisible by $a+b$, when n is an even integer.

Theorem 4. The expression a^n+b^n is never exactly divisible by $a-b$, whether n be even or odd.

As before, we have $a^n+b^n=(a-b)\times Q+R$ (identically).

Putting b for a in the identity, we have

$$b^n+b^n=(b-b)\times Q'+R=0\times Q'+R=R.$$

$$\therefore\ R=b^n+b^n=2b^n, \text{ but } R \text{ is not zero.}$$

Hence a^n+b^n is never divisible by $a-b$.

Examples 4.

Ex. 1. Prove that 13^8-6^8 is exactly divisible by 19.

$\because$ 8 is an even integer, $\therefore$ 13^8-6^8 is divisible by $(13+6)$ or 19.

Ex. 2. If n be a positive integer, show that $5^{3n}-3^{4n}$ is divisible by 44.

$$5^{3n}-3^{4n}=(5^3)^n-(3^4)^n=(125)^n-(81)^n.$$

Now, ∵ n is a positive integer, ∴ $(125)^n-(81)^n$ is divisible by $(125-81)$ or 44.

Ex. 3. If n be a positive even integer, prove that each of the last two digits in $4^{3n}-6^{2n}$ is zero and it is divisible by 100.

The given expression $=(4^3)^n-(6^2)^n=(64)^n-(36)^n$.

Now, ∵ n is a positive even integer,

∴ $(64)^n-(36)^n$ is divisible by $(64+36)$ or 100.

Again, since 100 is a factor of the given expression, each of its last two digits must be 0.

Ex. 4. Show that the last digit in 3^n+2^n, when n is an odd positive integer, must be 5.

∵ n is an odd positive integer,

∴ 3^n+2^n is divisible by $(3+2)$ or 5.

Hence the last digit of the expression must be either 5 or 0.

Now, since any power of an even number is even and any power of an odd number is odd, 3^n is odd and 2^n is even.

∴ Their sum must be odd. Hence the last digit in 3^n+2^n mus be 5 and not 0.

Ex. 5. Prove that a number is divisible by 9, when the sum of its digits is divisible by 9. [*P. U.* 1915]

Let m, l,......be the digits in the units' place, tens' place, etc.

Then the given number may be generally expressed as

$$p.10^n+q.10^{n-1}+r.10^{n-2}+\cdots+l.10+m.$$

Now, the remainder that is left, when this expression is divided by 9, i.e., $(10-1)$, is obtained by putting 1 for 10 in the expression.

$$\therefore \text{ The remainder}=p(1)^n+q(1)^{n-1}+\cdots+l.1+m$$
$$=p+q+\cdots+l+m.$$

Hence, if this remainder, *i.e.*, the sum of p, q, r,......, l, m be divisible by 9, the given number also will be divisible by 9.

[*Otherwise*] If the given number be of one digit, it must be 9 and so divisible by 9.

Suppose the number consists of two digits and let x be the digit in the units' place and y in the tens' place.

Then the number $=10y+x$.

$$\text{Now,}\quad (10y+x)\div 9=\frac{9y+(y+x)}{9}=y+\frac{y+x}{9}.$$

Hence, if $(y+x)$, *i.e.*, the sum of the digits be divisible by 9, then the number (*i.e.*, $10y+x$) is divisible by 9.

Similarly it can be shown that this holds good when the number consists of more than 2 digits.

Ex. 6. Show that if the difference between the sums of the digits in the odd and even places of a number is zero or divisible by 11, the number itself is divisible by 11.

Let the number consist of four digits and let x, y, z, p respectively be the first four digits.

Then the number $=1000x+100y+10z+p$.

$$\text{Now,}\quad \frac{1000x+100y+10z+p}{11}=90x+9y+\frac{10x+10z+y+p}{11}$$

$$=90x+9y+x+z+\frac{-x-z+y+p}{11}$$

$$=91x+9y+z+\frac{(y+p)-(x+z)}{11}.$$

Hence, it is evident that if $(y+p)-(x+z)$ be 0 or divisible by 11, the given number will also be divisible by 11.

Similarly it can be shown that it holds good when the number consists of more than 4 digits.

Ex. 7. Find the continued product of

$$(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8).$$

Let x be the product. Then, $x=(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$
Multiplying both sides by $a-b$, we have

$$\begin{aligned}(a-b)x &= (a^2-b^2)(a^2+b^2)(a^4+b^4)(a^8+b^8)\\ &= (a^4-b^4)(a^4+b^4)(a^8+b^8)\\ &= (a^8-b^8)(a^8+b^8)=a^{16}-b^{16},\end{aligned}$$

$$\therefore\ x=\frac{a^{16}-b^{16}}{a-b}=a^{15}+a^{14}b+a^{13}b^2+a^{12}b^3+\cdots+ab^{14}+b^{15}.$$

Ex. 8. If $2x^3+px^2-21x+q$ be an integral multiple of x^2+x-12, find the values of p and q.

Here the divisor $=x^2+x-12=(x-3)(x+4)$

Now, $\because$ $2x^3+px^2-21x+q$ is an integral multiple of x^2+x-12, *i.e.*, of $(x-3)(x+4)$, it must be exactly divisible by $(x-3)$ and $(x+4)$.

The remainders obtained by dividing $2x^3+px^2-21x+q$ by $(x-3)$ and $(x+4)$ will be 0 in each case.

Now, the remainder in the first case

$$=2(3)^3+p.3^2-21.3+q=54+9p-63+q=9p+q-9\,;$$

$\therefore\ 9p+q-9=0$, or, $9p+q=9$......(i)

Again the remainder in the second case

$$=2(-4)^3+p(-4)^2-21\times-4+q=-128+16p+84+q$$
$$=16p+q-44\,;$$

$\therefore\ 16p+q-44=0$, or, $16p+q=44$ ... (ii)

Solving (i) and (ii) we have $p=5$, $q=36$.

Ex. 9. If n is any positive integer, show that $(ab)^n-(bc)^n+(cd)^n-(da)^n$ is divisible by $ab-bc+cd-da$. [*M. U.* 1878]

$$ab-bc+cd-da=b(a-c)-d(a-c)=(a-c)(b-d).$$

Here the given expression is a rational integral function.

Putting c for a in the expression, we have

the given exp. $=(cb)^n-(bc)^n+(cd)^n-(dc)^n$

$$=(bc)^n-(bc)^n+(dc)^n-(dc)^n=0.$$

$\therefore$ the expression is divisible by $a-c$.

Similarly by putting d for b, it can be proved that the expression is divisible by $b-d$.

Hence the expression is divisible by the product of $(a-c)$ and $(b-d)$, *i.e.*, by $ab-bc+cd-da$.

Ex. 10. Show that $a^n(b-c)+b^n(c-a)+c^n(a-b)$ contains each of the factors $a-b$, $b-c$, $c-a$. [*P. U.* 1916]

$a-b$ will be a factor of the given expression if the expression becomes zero, when $a-b=0$, *i.e.*, when $a=b$.

Now putting b for a in the expression, we have

$$\text{the exp.}=b^n(b-c)+b^n(c-b)+c^n(b-b)$$
$$=b^n(b-c)-b^n(b-c)+c^n\times 0=0.$$

Hence $a-b$ is a factor of the given expression. Similarly it can be shown that $b-c$ and $c-a$ are also factors of the expression.

Ex. 11. For what value of m will $2x^3+7x^2-x+m$ and $3x^2+10x-11$ leave the same remainder when divided by $x+4$?

The two remainders, left by the division of the given expressions by $x+4$, can be obtained by putting $x=-4$ in them,

$\therefore$ The remainder from the first expression

$=2(-4)^3+7(-4)^2-(-4)+m=-128+112+4+m=-12+m$;

and the remainder from the second expression

$$=3(-4)^2+10(-4)-11=48-40-11=-3.$$

$\because$ The remainders are equal by the problem,

$$\therefore\ -12+m=-3, \quad \therefore\ m=9.$$

Exercise 3

1. If $f(a)\equiv 2a^3-3a^2+5a+6$, find the value of

(i) $f(2)$, (ii) $f(-1)$, (iii) $f(-3)$, (iv) $f(n)$.

2. Find, without actual division, the remainder in each of the following :—

(i) $(x^3+2x^2-3x+4)\div(x-3)$

(ii) $(2a^3-3a^2+5a+7)\div(a+2)$

(iii) $(5x^3+9x^2+10)\div(x+3)$

(iv) $(3a^4-2a^3+5a^2+a-36)\div(a-4)$

(v) $(4x^3+3x^2-5x-6)\div(2x+1)$

3. Show that the first expression is exactly divisible by the second ;—

(i) $x^3-6x^2+11x-6, x-2$ (ii) $x^{12}-1, x-1$

(iii) $2x^3+3x^2-5x+12, x+3$

(iv) $15x^3-4x^2+15x+6, 3x+1$

(v) $6x^3-13x^2y+8xy^2-3y^3, 2x-3y$

(vi) $a^{10}+b^{19}, a+b$.

4. Show that $625x^4-16$ is exactly divisible by $5x-2$.

[*P. U.* '21]

5. Show that $(ax+by)^3+(bx+ay)^3$ is divisible by $a+b$ and also by $x+y$.

6. For what value of m is $x^3+6x^2+10x+m$ exactly divisible by $x+3$?

7. If $x-2$ is a factor of $120x^3-167x^2-ax+56$, find the value of a. [*D. U.* 1926]

8. If $x^2-3ax+b^2$ is exactly divisible by $x-a$, show that $2a^2=b^2$.

9. If $2x^3-7x^2+5x+p$ and $3x^2-11x+17$ leave the same remainder when divided by $x-3$, find the value of p.

10. If n is a positive integer, show that 9^n-1 is always divisible by 8.

11. Show that $2x+3$ is a factor of $6x^2+7x-3$.

12. Prove that $3^{4n}-4^{3n}$ is divisible by 17, if n be a positive integer.

13. If n be a positive even integer, show that each of the last two digits in $2^{6n}-6^{2n}$ is zero.

14. If n be a positive integer, show that the last digit in $3^{2n+1}+2^{2n+1}$ is 5.

15. Find the continued product of $(x+y)(x^2+y^2)(x^4+y^4)$.

16. Find the condition that $pa^n+qa^{n-1}+ra^{n-2}+\cdots+la+m$ is divisible by $a-1$.

17. Find the condition that x^2+px+q and $x^2+p'x+q'$ may have a common factor of the form $x+a$.

18. If $x+p$ be the H. C. F. of x^2+ax+b and $x^2+a'x+b'$, shew that $p=\dfrac{b-b'}{a-a'}$.

19. Show that $a-b$, $b-c$ and $c-a$ are factors of
$$a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3.$$

20. If ax^3+bx^2+cx+d be divided by $x-p$ until the remainder is independent of x, find the remainder without actually performing the division.

Use this theorem to prove that $x^6+ax^5+cx^3+dx^2-1$ is divisible by $x+1$, if $a+c=d$. [*A. U.* 1913]

21. What must be the form of m so that a^m-x^m may have both a^n+x^n and a^n-x^n for divisors, n being any positive integer. [*M. U.* 1875]

22. If n be a positive integer, shew that $x^n(y-z)+y^n(z-x)+z^n(x-y)$ is exactly divisible by $(x-y)(y-z)(z-x)$.

23. If $x-p$ be the H. C. F. of ax^2+bx+c and cx^2+ax+b, prove that $a^3+b^3+c^3=3abc$.

SIMULTANEOUS EQUATIONS

Examples 5

Ex. 1. Solve $\left.\begin{array}{l}7x-3y=31\\ 9x-5y=41\end{array}\right\}$ [*D. B.* 1934]

$$\begin{array}{l}7x-3y=31\cdots\cdots\cdots\cdots(1)\\ 9x-5y=41\cdots\cdots\cdots\cdots(2)\end{array}$$

Multiplying (1) by 5 and (2) by 3 we have

$$\begin{array}{r}35x-15y=155\\ \underline{27x-15y=123}\end{array}$$

$$(\text{subtracting})\ \ 8x=32, \qquad \therefore\ \ x=4.$$

Now putting $x=4$ in (1) we have $28-3y=31$, or, $-3y=3$,

$\therefore\ \ y=-1.\ \ \therefore$ the solution is $x=4,\ y=-1.$

Ex. 2. Solve $x+2y=3=4x-y.$ [*C. U.* 1917]

Here $x+2y=3\cdots\cdots\cdots(1)$ and $4x-y=3\cdots\cdots\cdots(2)$

Multiplying (2) by 2, we get $8x-2y=6$

$$\underline{\text{and (1) is}\qquad\qquad x+2y=3}$$

$$\therefore\ (\text{adding})\qquad 9x=9\qquad \therefore\ \ x=1.$$

Now, from (1) we have $1+2y=3$, or, $2y=2$, $\therefore\ \ y=1.$

$\therefore\ \ x=1,\ y=1.$

Ex. 3. Solve $\left.\begin{array}{l}x+5y=36\\[2ex] \dfrac{x+y}{x-y}=\dfrac{5}{3}\end{array}\right\}$ [*C. U.* 1912]

$x+5y=36\ldots\ldots\ldots(1),\ \dfrac{x+y}{x-y}=\dfrac{5}{3}\ldots\ldots(2)$

From (2) we have $5x-5y=3x+3y$

or, $2x-8y=0$, or, $x-4y=0\cdots(3)$

$$\begin{array}{l}\text{Now, } x+5y=36\cdots(1)\\ \underline{\text{and}\ \ \ x-4y=\ 0\cdots(3)}\end{array}$$

$$\therefore\ (\text{subtracting})\ 9y=36, \qquad \therefore\ \ y=4.$$

$\therefore$ from (1), $x+20=36$, or, $x=36-20=16.$

$\therefore\ \ x=16,\ \ y=4.$

Ex. 4. Solve $\left.\begin{array}{l}\dfrac{2}{x}+\dfrac{3}{y}=2\ldots\ldots(1)\\ \dfrac{5}{x}+\dfrac{10}{y}=5\frac{5}{6}\ldots\ldots(2)\end{array}\right\}$

Multiplying (1) by 5 and (2) by 2 we have

$$\frac{10}{x}+\frac{15}{y}=10$$

$$\text{and } \frac{10}{x}+\frac{20}{y}=\frac{35}{3}$$

(subtracting) $-\dfrac{5}{y}=-\dfrac{5}{3}$, or, $\dfrac{5}{y}=\dfrac{5}{3}$, $\therefore\ y=3$.

Now from (1), $\dfrac{2}{x}+\dfrac{3}{3}=2$, or, $\dfrac{2}{x}=2-1=1$, $\therefore\ x=2$.

$$\therefore\ x=2,\ y=3.$$

Ex. 5. Solve $\dfrac{x+y}{xy}=2$, $\dfrac{x-y}{xy}=1$. [*D. B.* '31]

$$\frac{x+y}{xy}=2\ldots\ldots(1),\quad \frac{x-y}{xy}=1\ldots\ldots(2).$$

From (1), $\dfrac{x}{xy}+\dfrac{y}{xy}=2$, or, $\dfrac{1}{y}+\dfrac{1}{x}=2\ldots\ldots(3)$.

From (2), $\dfrac{x}{xy}-\dfrac{y}{xy}=1$, or, $\dfrac{1}{y}-\dfrac{1}{x}=1\ldots\ldots(4)$.

Adding (3) and (4) we have $\dfrac{2}{y}=3$,

or, $3y=2$, $\therefore\ y=\dfrac{2}{3}$.

Again, subtracting (4) from (3) we have

$$\frac{2}{x}=1,\quad \therefore\ x=2.$$

Hence. $x=2$, $y=\dfrac{2}{3}$.

Ex. 6. Solve $23x+17y=63\ldots\ldots\ldots\ldots(1)$

$$\frac{17x+23y=57\ldots\ldots\ldots\ldots(2)}{}$$

(Adding) $40x+40y=120$

or, $x+y=3\ldots\ldots(3)$ [dividing both sides by 40]

Again, from $(1)-(2)$ we have $6x-6y=6$,

or, $x-y=1\ldots\ldots\ldots\ldots(4)$.

Adding (3) and (4) we have $2x=4$, $\therefore$ $x=2$;

and from $(3)-(4)$,, ,, $2y=2$, $\therefore$ $y=1$.

$\therefore$ $x=2$, $y=1$.

Ex. 7. Solve $\left.\begin{array}{l}\dfrac{x+2}{7}+\dfrac{y-x}{4}=2x-8\ldots\ldots(1)\\ \dfrac{2y-3x}{3}+2y=3x+4\ldots\ldots(2)\end{array}\right\}$ [*P. U.* 1892]

Multiplying (1) by 28 we have

$4x+8+7y-7x=56x-224$, or, $-59x+7y=-232\ldots\ldots(3)$

Multiplying (2) by 3, we have $2y-3x+6y=9x+12$,

or, $8y-12x=12$, or, $-3x+2y=3\ldots\ldots(4)$.

Again, multiplying (3) by 2 and (4) by 7 we have

$$\begin{array}{r}-118x+14y=-464\\ -\ \ 21x+14y=\ \ \ \ 21\\ \hline\end{array}$$

(Subtracting) $-\ 97x=-485$, $\therefore$ $x=\frac{-485}{-97}=5$.

Now from (4), $-15+2y=3$, or, $2y=18$, $\therefore$ $y=9$.

$\therefore$ $x=5$, $y=9$.

Ex. 8. Solve $\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{y}{b}=a+b\ldots\ldots(1).\\ \dfrac{x}{a^2}+\dfrac{y}{b^2}=2\ldots\ldots(2).\end{array}\right\}$

Multiplying (1) by a and (2) by a^2 we have

$$x+\frac{a}{b}y=a^2+ab\ldots\ldots(3)$$

$$\text{and } x+\frac{a^2}{b^2}y=2a^2\ldots\ldots\ldots(4)$$

$\therefore$ (subtracting) $\frac{ab-a^2}{b^2}\cdot y=ab-a^2$,

$$\therefore\ y=\frac{(ab-a^2)\times b^2}{(ab-a^2)}=b^2.$$

From (3), we have $x+\frac{a}{b}\times b^2=a^2+ab$ [putting b^2 for y]

or, $x+ab=a^2+ab$, $\therefore$ $x=a^2$. $\therefore$ $x=a^2$, $y=b^2$.

Ex. 9. Solve $\left.\begin{aligned} x+2y+3z&=20 \ldots\ldots(1)\\ 2x+3y-5z&=-7 \ldots\ldots(2)\\ 4x-5y+7z&=21 \ldots\ldots(3)\end{aligned}\right\}$ [*C. U.* 1898]

Multiplying (1) by 2 we have

$$2x+4y+6z=40$$

$$\text{and (2) is } 2x+3y-5z=-7$$

$\therefore$ (subtracting) $y+11z=47\ldots\ldots(4)$

Again, multiplying (1) by 4 we have

$$4x+8y+12z=80$$

$$\text{and (3) is } 4x-5y+7z=21$$

$\therefore$ (subtracting) $13y+5z=59\ldots\ldots(5)$

Now, multiplying (4) by 13, we have

$$13y+143z=611$$

$$\text{and (5) is } 13y+5z=59$$

$\therefore$ (subtracting) $138z=552$, $\therefore$ $z=\frac{552}{138}=4$.

From (4), $y+44=47$, $\therefore$ $y=47-44=3$.

From (1), $x+6+12=20$, $\therefore$ $x=20-18=2$.

$\therefore$ $x=2$, $y=3$, $z=4$.

Ex. 10. Solve $\left.\begin{array}{l} x+y=5 \\ y+z=7 \\ z+x=6 \end{array}\right\}$ Here $x+y=5 \ldots\ldots(1)$

$y+z=7 \ldots\ldots(2)$

$z+x=6 \ldots \quad (3)$

$\therefore$ (Adding) $2(x+y+z)=18,$

$\therefore \quad x+y+z=9 \ldots\ldots(4)$

Now, from $(4)-(1)$ we have $z=9-5=4,$

,, $(4)-(2)$,, ,, $x=9-7=2,$

and ,, $(4)-(3)$,, ,, $x=9-6=3.$ $\therefore$ $x=2, y=3, z=4.$

Ex. 11. Solve
$$\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=6 \\ \frac{1}{y}+\frac{1}{z}=7 \\ \frac{1}{z}+\frac{1}{x}=5 \end{array}\right\} \begin{array}{l} \cdots\cdots\cdots(1) \\ \cdots\cdots\cdots(2) \\ \cdots\cdots\cdots(3) \end{array}$$

(Adding) $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=18,$

$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9 \cdots\cdots(4)$

Now from $(4)-(1)$, $\frac{1}{z}=3$, or, $3z=1$, $\therefore \quad z=\frac{1}{3}$

,, $(4)-(2)$, $\frac{1}{x}=2$, or, $2x=1$, $\therefore \quad x=\frac{1}{2}$

and, $(4)-(3)$, $\frac{1}{y}=4$, or, $4y=1$, $\therefore \quad y=\frac{1}{4}$

Hence, $x=\frac{1}{2}$, $y=\frac{1}{4}$, $z=\frac{1}{3}$.

Ex. 12 Solve $\frac{x+y}{xy}=\frac{3}{2} \cdots(1)$, $\frac{y+z}{yz}=\frac{5}{6} \cdots(2)$, $\frac{z+x}{zx}=1\frac{1}{3} \cdots(3)$

From (1) we have $\frac{x}{xy}+\frac{y}{xy}=\frac{3}{2}$, or, $\frac{1}{y}+\frac{1}{x}=\frac{3}{2}$……(4)

,, (2) ,, ,, $\frac{y}{yz}+\frac{z}{yz}=\frac{5}{6}$, or, $\frac{1}{z}+\frac{1}{y}=\frac{5}{6}$……(5)

,, (3) ,, ,, $\frac{z}{zx}+\frac{x}{zx}=\frac{4}{3}$, or, $\frac{1}{x}+\frac{1}{z}=\frac{4}{3}$……(9)

(Adding) $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{11}{3}$,

$\therefore\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{6}$……(7).

Now from (7)−(4) we have $\frac{1}{z}=\frac{11}{6}-\frac{3}{2}=\frac{1}{3}$, $\therefore\ z=3$

,, (7)−(5) ,, ,, $\frac{1}{x}=\frac{11}{6}-\frac{5}{6}=1$, $\therefore\ x=1$

and ,, (7)−(6) ,, ,, $\frac{1}{y}=\frac{11}{6}-\frac{4}{3}=\frac{1}{2}$, $\therefore\ y=2$

$\therefore\ x=1,\ y=2,\ z=3.$

Ex. 13. Solve

$$\left.\begin{array}{l} 3x+4y=5xy\cdots\cdots(1) \\ 3y+5z=6yz\cdots\cdots(2) \\ 2z+3x=2zx\cdots\ (3) \end{array}\right\} \qquad [\text{ H. S. '60 }]$$

Dividing (1) by xy we have

$$\frac{3x}{xy}+\frac{4y}{xy}=5, \quad \text{or,} \quad \frac{3}{y}+\frac{4}{x}=5 \quad \cdots(4)$$

Similarly, dividing (2) by yz and (3) by zx we have

$$\frac{3}{z}+\frac{5}{y}=6\cdots\cdots(5)$$

and $\frac{2}{x}+\frac{3}{z}=2\cdots\cdot(6)$

Now from, (5)−(6) we have

$$\frac{5}{y}-\frac{2}{x}=4\cdots\quad(7)$$

Again from $(4)\times 1$ and $(7)\times 2$ we have

$$\frac{3}{y}+\frac{4}{x}=5$$

$$\text{and }\frac{10}{y}-\frac{4}{x}=8$$

$$\text{(Adding) }\frac{13}{y}=13,\quad\text{or,}\quad 13y=13,\quad\therefore\quad y=1.$$

$\therefore$ from (4) we have $3+\frac{4}{x}=5$, or, $\frac{4}{x}=2$,

or, $2x=4$, $\therefore\ x=2$.

From (5) we have $\frac{3}{z}+5=6$, or, $\frac{3}{z}=1$, $\therefore\ z=3$.

$$\therefore\quad x=2,\ y=1,\ z=3.$$

Ex. 14. Solve $\frac{xy}{x+y}=\frac{6}{5}$, $\frac{yz}{y+z}=\frac{12}{7}$ and $\frac{zx}{z+x}=\frac{4}{3}$.

[By inverting we have $\frac{x+y}{xy}=\frac{5}{6}$, $\frac{y+z}{yz}=\frac{7}{12}$ and $\frac{z+x}{zx}=\frac{3}{4}$.

Now work out as in Ex. 12. Ans.: $x=2$, $y=3$, $z=4$]

Ex. 15. Solve $\frac{x+y}{xy}=\frac{y+z}{yz}=\frac{z+x}{zx}=\frac{2}{5}$.

$$\frac{x+y}{xy}=\frac{2}{5},\ \text{or,}\ \frac{1}{y}+\frac{1}{x}=\frac{2}{5}\cdots\cdots(1)$$

$$\frac{y+z}{yz}=\frac{2}{5},\ \text{or,}\ \frac{1}{z}+\frac{1}{y}=\frac{2}{5}\cdots\cdots(2)$$

$$\frac{z+x}{zx}=\frac{2}{5},\ \text{or,}\ \frac{1}{x}+\frac{1}{z}=\frac{2}{5}\cdots\cdots(3)$$

$$\text{(Adding),}\quad 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{6}{5},\quad\therefore\quad\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{5}\cdots\cdots(4)$$

From (4) − (1) we have $\frac{1}{z}=\frac{1}{5}$, $\therefore\ z=5$;

" (4) − (2) " " $\frac{1}{x}=\frac{1}{5}$, $\therefore\ x=5$

" (4) − (3) " " $\frac{1}{y}=\frac{1}{5}$, $\therefore\ y=5$

$\therefore\ x=y=z=5.$

Ex. 16. Solve $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $a^2x+b^2y+c^2z=a^3+b^3+c^3$.

Here, $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (suppose)

Then $x=ak,\ y=bk,\ z=ck$......(A)

Now, from the equation $a^2x+b^2y+c^2z=a^3+b^3+c^3$

we have $a^3k+b^3k+c^3k=a^3+b^3+c^3$,

or, $(a^3+b^3+c^3)k=a^3+b^3+c^3$,

$$\therefore\ k=\frac{a^3+b^3+c^3}{a^3+b^3+c^3}=1.$$

Hence from (A) we have $x=a,\ y=b,\ z=c$.

Ex. 17. Solve $a(x+y)=b(x-y)=2ab$. [*C. U.* 1930]

Here, $a(x+y)=2ab$......(1)

and $b(x-y)=2ab$......(2)

From (1) we get $x+y=2b$ (dividing both sides by a)......(3)

" (2) " " $x-y=2a$ (" " " b)

$\therefore$ (Adding) $2x=2(a+b)$, $\therefore\ x=a+b$.

Now, from (3), $a+b+y=2b$, $\therefore\ y=2b-b-a=b-a$.

Hence, $x=a+b,\ y=b-a$.

Ex. 18. Solve $xy=12$...(1), $yz=20$...(2) and $zx=15$...(3).

Multiplying (1), (2) and (3) we have

$$x^2y^2z^2=12\times20\times15=3600,$$

$$\therefore\ xyz=\pm\sqrt{3600}=\pm60\text{......(4)}.$$

Now, from (4)÷(1) we have $\frac{xyz}{xy}=\pm\frac{60}{12}$, or, $z=\pm5$.

From, (4)÷(2), $x=\pm3$ and from (4)÷(3), $y=\pm4$.

$\therefore\ x=\pm3,\ y=\pm4,\ z=\pm5$.

15. Method of cross-multiplication

When of three given equations. two are equal to 0, they may be easily solved by the method of cross-multiplication.

From the two equations which are equal to 0, we may write as follows :—

$$\frac{x}{(\text{co-eff. of } y\times\text{co-eff. of } z)-(\text{co-eff. of } z\times\text{co-eff. of } y)}$$

$$=\frac{y}{(\text{co-eff. of } z\times\text{co-eff. of } x)-(\text{co-eff. of } x\times\text{co-eff. of } z)}$$

$$=\frac{z}{(\text{co-eff. of } x\times\text{co-eff. of } y)-(\text{co-eff. of } y\times\text{co-eff. of } x)};$$

it is to be remembered that the coefficient mentioned first in each bracket should be taken from the first equation (*i.e.*, from the upper one, if one is put below the other), and the coefficient multiplied with it should be taken from the second equation.

The proof of the Rule of cross-multiplication is given below ;—

Take the equations

$ax+by+cz=0$......(1), and $a_1x+b_1y+c_1z=0$......(2).

Multiplying (1) by c_1 and (2) by c we have

$$ac_1x+bc_1y+cc_1z=0\text{......(3)}$$

and $$a_1cx+b_1cy+cc_1z=0\text{......(4)}$$

From (4)−(3) we have $(a_1c-ac_1)x+(b_1c-bc_1)y=0$,

or, $(a_1c-ac_1)x=(bc_1-b_1c)y$

$$\therefore\ \frac{x}{bc_1-b_1c}=\frac{y}{a_1c-ac_1}\text{......(5)}$$

Again, multiplying (1) by a_1 and (2) by a we have

$$aa_1x+a_1by+a_1cz=0\cdots\cdots(6)$$

and $$aa_1x+ab_1y+ac_1z=0\cdots\cdots(7)$$

Now, from (7) − (6) we have

$$(ab_1-a_1b)y+(ac_1-a_1c)z=0,$$

or, $$(ab_1-a_1b)y=(a_1c-ac_1)z$$

$$\therefore\quad \frac{x}{a_1c-ac_1}=\frac{y}{ab_1-a_1b}\cdots\quad\cdots(8)$$

$\therefore$ from (5) and (8) we have

$$\frac{y}{bc_1-b_1c}=\frac{z}{a_1c-ac_1}=\frac{z}{ab_1-a_1b}.$$

Ex. 1. Solve $\left.\begin{aligned} x-2y+\ z&=\ 0\ldots\ldots(1)\\ 3x+6y-5z&=\ 0\ldots\ldots(2)\\ 2x+3y+4z&=20\ldots\ldots(3)\end{aligned}\right\}$

From (1) and (2) we have by the rule of cross-multiplication

$$\frac{x}{(-2\times-5)-(1\times6)}=\frac{y}{(1\times3)-(1\times-5)}=\frac{z}{(1\times6)-(-2\times3)}$$

or, $\dfrac{x}{10-6}=\dfrac{y}{3+5}=\dfrac{z}{6+6}$, or, $\dfrac{x}{4}=\dfrac{y}{8}=\dfrac{z}{12}$,

or, $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}=k$ (suppose)

Then $x=k,\ y=2k,\ z=3k\quad\cdots\cdots(A)$

Now, from (3) we have $2k+6k+12k=20$,

or, $20k=20,\qquad k=1.$

$\therefore$ Putting $k=1$ in (A) we have $x=1,\ y=2,\ z=3$.

Ex. 2. Solve $\left.\begin{aligned} x+\ y+\ z&=0\cdots\cdots(1)\\ ax+by+cz&=0\cdots\cdots(2)\\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}&=0\cdots\cdots(3)\end{aligned}\right\}$

By the rule of cross-multiplication we have from (1) and (2)

$$\frac{x}{c-b}=\frac{y}{a-c}=\frac{z}{b-a}, \text{ or, } \frac{x}{-(b-c)}=\frac{y}{-(c-a)}=\frac{z}{-(a-b)}$$

$$\text{or, } \frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}=k \text{ (suppose)}$$

Then $x=k(b-c)$, $y=k(c-a)$, $z=k(a-b)$......(4).

Now, from (3) we have $\frac{k(b-c)}{b-c}+\frac{k(c-a)}{c-a}+\frac{k(a-b)}{a-b}=3$,

or, $k+k+k=3$, or, $3k=3$, $\therefore$ $k=1$.

$\therefore$ from (4) we have $x=b-c$, $y=c-a$, $z=a-b$.

Ex. 3. Solve

$$\left.\begin{aligned} x+y+z=0 &\cdots\cdots(1)\\ bcx+cay+abz=0 &\cdots\cdots(2)\\ ax+by+cz+(b-c)(c-a)(a-b)=0 &\cdots\cdots(3) \end{aligned}\right\}$$

From (3) we have $ax+by+cz=-(b-c)(c-a)(a-b)$......(4).
From (1) and (2) by the rule of cross-multiplication

we have $\frac{x}{ab-ca}=\frac{y}{bc-ab}=\frac{z}{ca-bc}$,

$$\text{or, } \frac{x}{a(b-c)}=\frac{y}{b(c-a)}=\frac{z}{c(a-b)}=k \text{ (suppose)}$$

Then $x=a(b-c)k$, $y=b(c-a)k$, $z=c(a-b)k\cdots\cdots(A)$

Now, putting these values of x, y, z in (4) we have

$$a^2(b-c)k+b^2(c-a)k+c^2(a-b)k=-(b-c)(c-a)(a-b)$$

or, $k\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}=-(b-c)(c-a)(a-b)$

or, $-(a-b)(b-c)(c-a)k=-(b-c)(c-a)(a-b)$

$\therefore$ $k=1$.

$\therefore$ from (A) we have $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

Ex. 4. Solve

$$\left.\begin{aligned} 24x=15y=40z&\ldots\ldots(1)\\ \text{and } 3x+y+2z=29 &\ldots\ldots(2) \end{aligned}\right\}$$

Dividing (1) by 120, the L. C. M. of 24, 15 and 40, we have

$$\frac{x}{5}=\frac{y}{8}=\frac{z}{3}=k \text{ (suppose)}$$

Then $x=5k$, $y=8k$, $z=3k\cdots\cdots(A)$

Now, from (2) we have $15k+8k+6k=29$,

or, $29k=29$, $\therefore\ k=1$.

$\therefore$ From (A) we have $x=5,\ y=8,\ z=3$.

Ex. 5. Solve
$$\left.\begin{aligned} x+y+z=a+b+c\ldots\ldots\ldots\ldots(1)\\ ax+by+cz=a^2+b^2+c^2\ldots\ (2)\\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3\ \cdots\cdots(3)\end{aligned}\right\}$$

From (1) we have $(x-a)+(y-b)+(z-c)=0\ldots(4)$

From (2) ,, ,, $(ax-a^2)+(by-b^2)+(cz-c^2)=0$,

or, $a(x-a)+b(y-b)+c(z-c)=0\ldots(5)$.

By the rule of cross-multiplication we have from (4) and (5)

$$\frac{x-a}{c-b}=\frac{y-b}{a-c}=\frac{z-c}{b-a}=k \text{ (suppose)}$$

Then $x-a=k(c-b),\ y-b=k(a-c),\ z-c=k(b-a)\ldots(A)$

Now, from (3) we have by transposition

$$\frac{x}{a}-1+\frac{y}{b}-1+\frac{z}{c}-1=0, \quad \text{or, } \frac{x-a}{a}+\frac{y-b}{b}+\frac{z-c}{c}=0,$$

or, $$\frac{k(c-b)}{a}+\frac{k(a-c)}{b}+\frac{k(b-a)}{c}=0$$

(putting the values of $x-a,\ y-b,\ z-c$)

or, $$k\left(\frac{c-b}{a}+\frac{a-c}{b}+\frac{b-a}{c}\right)=0, \quad \therefore\ k=0.$$

Hence from (A) we get,
$$\left.\begin{aligned} x-a=0,\ \therefore\ x=a\\ y-b=0,\ \therefore\ y=b\\ z-c=0,\ \therefore\ z=c\end{aligned}\right\} \text{ (Ans.)}$$

Ex. 6. Solve
$$\left.\begin{aligned} \frac{2}{x}+\frac{5}{y}-\frac{3}{z}=0\ \cdots\cdots\cdots(1)\\ \frac{3}{x}-\frac{12}{y}+\frac{2}{z}=0\cdots\cdots\cdots(2)\\ 4x+y+3z=4\cdots\cdots\cdots(3)\end{aligned}\right\}$$

From (1) we have $2.\frac{1}{x}+5.\frac{1}{y}-3.\frac{1}{z}=0$

,, (2) ,, ,, $3.\frac{1}{x}-12.\frac{1}{y}+2.\frac{1}{z}=0,$

Now by the rule of cross-multiplication we have

$$\frac{\frac{1}{x}}{(5\times 2)-(-3\times -12)}=\frac{\frac{1}{y}}{(-3\times 3)-(2\times 2)}=\frac{\frac{1}{z}}{(2\times -12)-(5\times 3)}$$

or, $\frac{\frac{1}{x}}{-26}=\frac{\frac{1}{y}}{-13}=\frac{\frac{1}{z}}{-39}$ or, $\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{1}=\frac{\frac{1}{z}}{3}=k$ (suppose)

$\therefore\ \frac{1}{x}=2k,\ \frac{1}{y}=k,\ \frac{1}{z}=3k,$ or, $x=\frac{1}{2k},\ y=\frac{1}{k},\ z=\frac{1}{3k}$...(A)

Now, from (3) we have $4.\frac{1}{2k}+\frac{1}{k}+3.\frac{1}{3k}=4,$

or, $\frac{2}{k}+\frac{1}{k}+\frac{1}{k}=4,$ or, $\frac{4}{k}=4,$ $\therefore\ k=1.$

$\therefore$ From (A) we have $x=\frac{1}{2},\ y=1,\ z=\frac{1}{3}.$

[*Miscellaneous Examples*]

Ex. 1. Solve $x(x+y+z)=6$...(1) $y(x+y+z)=12$...(2) and $z(x+y+z)=18$...(3).

Adding (1), (2) and (3) we have $(x+y+z)(x+y+z)=36,$

or, $(x+y+z)^2=36,$ $\therefore\ x+y+z=\pm 6$......(4)

$$\left.\begin{array}{l}\text{Dividing (1) by (4) we have } x=\pm 1\\ \text{,,} \quad \text{(2) by (4) ,, ,, } y=\pm 2\\ \text{,,} \quad \text{(3) by (4) ,, ,, } z=\pm 3\end{array}\right\}\quad \text{(Ans.)}$$

Ex. 2. Solve
$$\left.\begin{array}{l}\frac{1}{x-1}+\frac{1}{y-2}=3\cdots\cdots(1)\\ \frac{2}{x-1}+\frac{3}{y-2}=5\cdots\cdots(2)\end{array}\right\}\quad [\ A.\ U.\ \text{'42}\ ]$$

From (1)×2 we have

$$\frac{2}{x-1}+\frac{2}{y-2}=6$$

and (2) is $\dfrac{2}{x-1}+\dfrac{3}{y-2}=5$

(Subtracting) $-\dfrac{1}{y-2}=1$, or, $y-2=-1$, $\therefore\ y=2-1=1$.

Now from (1) we have $\dfrac{1}{x-1}+\dfrac{1}{1-2}=3$, or, $\dfrac{1}{x-1}-1=3$,

or, $\dfrac{1}{x-1}=4$, or, $4x-4=1$, or, $4x=5$, $\therefore\ x=\frac{5}{4}=1\frac{1}{4}$.

$\therefore\ x=1\frac{1}{4},\ y=1$.

Ex. 3. In the cyclic quadrilateral $ABCD$, $\angle A=(2x+13)$ degrees, $\angle B=(2y-18)$ degrees, $\angle C=(y+31)$ degrees, $\angle D=(3x-29)$ degrees. Find the values of x and y. [*P. U.* 1932]

$\because$ two opposite angles of a cyclic quadrilateral = 2 rt. angles,

$\therefore\ \angle A+\angle C=180^\circ$ and $\angle B+\angle D=180^\circ$.

$\therefore\ 2x+13+y+31=180 \cdots\cdots (1)$

and $3x-29+2y-18=180 \cdots\cdots (2)$

From (1) we have $2x+y=136 \cdots\cdots (3)$

and from (2) we have $3x+2y=227 \cdots\cdots (4)$.

Now, solving (3) and (4) we get $x=45$, $y=46$.

$\therefore\ x=45^\circ,\ y=46^\circ$:

Ex. 4. Solve $\left.\begin{aligned} a(y+z)&=yz \ldots\ldots (1)\\ b(z+x)&=zx \ldots\ldots (2)\\ c(x+y)&=xy \ldots\ldots (3)\end{aligned}\right\}$

From (1) we have $\dfrac{y+z}{yz}=\dfrac{1}{a}$, or, $\dfrac{1}{z}+\dfrac{1}{y}=\dfrac{1}{a} \ldots\ldots (4)$

" (2) " " $\dfrac{z+x}{zx}=\dfrac{1}{b}$, or, $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{b} \ldots\ldots (1)$

" (3) " " $\dfrac{x+y}{xy}=\dfrac{1}{c}$, or, $\dfrac{1}{y}+\dfrac{1}{x}=\dfrac{1}{c} \ldots\ldots (6)$

$\therefore$ Adding we have $2\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

$$\therefore\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdots\cdots(7)$$

$\therefore$ From (7)−(4) we have $\frac{1}{x}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{ac+ab-bc}{2abc}$

$$\therefore\ x=\frac{2abc}{ab+ac-bc};$$

" (7)−(5) " " $\frac{1}{y}=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$, $\therefore\ y=\frac{2abc}{ab+bc-ac}$,

and from (7)−(6) " " $\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$, $\therefore\ z=\frac{2abc}{bc+ac-ab}$.

Ex. 5. Eliminate t from the equations

$$x=t+\frac{1}{t},\quad y^2=t^2+\frac{1}{t^2}.$$ [*D. B.* '32]

Now, $x^2=\left(t+\frac{1}{t}\right)^2=t^2+\frac{1}{t^2}+2\cdot t\cdot\frac{1}{t}=t^2+\frac{1}{t^2}+2=y^2+2\cdot$

$\therefore$ eliminating t from the equations we have $x^2-y^2=2$.

Exercise 4

Solve :—

1. $\left.\begin{aligned}3x+5y&=69\\x-2y&=1\end{aligned}\right\}$ [*C. U.* '19]

2. $\left.\begin{aligned}6x-7y&=16\\9x-5y&=35\end{aligned}\right\}$ [*C. U.* '22]

3. $\left.\begin{aligned}\frac{x}{3}-\frac{2}{y}&=1\\\frac{x}{4}+\frac{3}{y}&=3\end{aligned}\right\}$ [*A. U.* '23]

4. $\left.\begin{aligned}x+y+z&=1\\2x+3y+z&=4\\4x+9y+z&=16\end{aligned}\right\}$ [*C. U.* '11]

5. (*a*) $\left.\begin{aligned}\frac{4}{x}+\frac{10}{y}&=2\\\frac{3}{x}+\frac{2}{y}&=\frac{19}{20}\end{aligned}\right\}$

5. (*b*) $\left.\begin{aligned}3x+4y&=11\\5x-2y&=1\end{aligned}\right\}$ [*W. B. S. F.* '53]

6. $\frac{2x+2y-3}{5}=\frac{3x-7y+4}{6}=\frac{8y-x+2}{7}$ [*C. U.* '14]

7. $\left.\begin{aligned}y+z&=6\\z+x&=4\\x+y&=2\end{aligned}\right\}$ [*C. U.* '18]

8. $\left.\begin{aligned}6y-x&=1\\\frac{x+y}{x-y}&=\frac{3}{2}\end{aligned}\right\}$ [*C. U.* '31]

9. $\dfrac{x+y}{xy}=5,\quad \dfrac{x-y}{xy}=9$ [*C. U.* '32]

10. $\left.\begin{aligned}\frac{5}{x}+3y&=8\\ \frac{4}{x}-10y&=56\end{aligned}\right\}$ [*D. B.* '29]

11. $\left.\begin{aligned}23x-24y&=21\\ 25x-16y&=43\end{aligned}\right\}$ [*D. B.* '36]

12. $\dfrac{x+y}{xy}=\dfrac{y+z}{yz}=\dfrac{z+x}{zx}=\dfrac{2}{3}$

13. $\left.\begin{aligned}x-2y+z&=0\\ 9x-8y+3z&=0\\ 2x+3y+5z&=36\end{aligned}\right\}$

14. $\left.\begin{aligned}6x=8y&=10z\\ 3x-4y+z&=12\end{aligned}\right\}$

15. $xy=24,\ yz=42,\ zx=28.$

16. $\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=6a\\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=10a\\ \frac{1}{z}+\frac{1}{x}-\frac{1}{y}&=-2a\end{aligned}\right)$

16. (*a*) $\left.\begin{aligned}2x-5y-3z&=0\\ 3x+3y-z&=0\\ -x+2y+5z&=11\end{aligned}\right\}$

17. $\dfrac{7x+y}{5y-z}=\dfrac{11y-z}{6x-z}=\dfrac{5x-y}{z+1}=2.$

18. $\left.\begin{aligned}&\frac{x}{a}=\frac{y}{b}=\frac{z}{c},\\ &ax+by+cz=a^2+b^2+c^2\end{aligned}\right\}$

19. $\left.\begin{aligned}&ax+by+cz=0\\ &a^2x+b^2y+c^2z=0\\ &x+y+z+(a-b)(b-c)(c-a)=0\end{aligned}\right\}$

20. $\left.\begin{aligned}&\frac{x}{4}=\frac{y}{3}=\frac{z}{5}\\ &4x+3y+z=30\end{aligned}\right\}$

21. $\left.\begin{aligned}x+y&=3xy\\ y+z&=5yz\\ z+x&=4zx\end{aligned}\right\}$

22. $\left.\begin{aligned}\frac{3}{x}+\frac{2}{y}-\frac{3}{z}&=0\\ \frac{5}{x}-\frac{6}{y}+\frac{2}{z}&=0\\ 2x+3y+4z&=6\end{aligned}\right\}$ [*D. B.* '33]

23. $\dfrac{x+ca}{b(c+a)}=\dfrac{y+bc}{a(b+c)}=\dfrac{z+ab}{c(a+b)}=\dfrac{x+y+z}{ab+bc+ca}$

[Hints :—From the first three by addendo we have

$$\text{Each}=\frac{x+y+z+ab+ac+bc}{2(ab+bc+ca)}=\frac{x+y+z}{ab+bc+ca}$$

$$\therefore \quad \text{Each} = \frac{x+y+z+ab+bc+ca-x-y-z}{2(ab+bc+ca)-(ab+bc+ca)} = 1.$$

Now solve taking each $=1$]

24. $x(y+z)=5,\ y(z+x)=8,\ z(x+y)=9.$

25. $ax+by=1,\ bx+ay=\dfrac{(a+b)^2}{a^2+b^2}-1.$ [*D. B.* '51]

[Hints :—Adding and subtracting the equations we have $x+y=\dfrac{a+b}{a^2+b^2}$ and $x-y=\dfrac{a-b}{a^2+b^2}\cdot$ Again add and subtract......]

26. $\dfrac{1}{x}+\dfrac{1}{y}=3,\ y+z=5yz,\ z+x=4xz.$ [*A. U.* '43]

27. $\left.\begin{aligned} ax+by&=c \\ bx+ay&=1+c \end{aligned}\right\}$ [*E. B. S. B.* '52]

28. $\dfrac{x}{a}+\dfrac{y}{b}=1,\quad \dfrac{y}{b}+\dfrac{z}{c}=1,\quad \dfrac{z}{c}+\dfrac{x}{a}=1.$ [*U. U.* '48]

$\left[\text{Hints :—Adding the equations, } 2\left(\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right)=3, \quad \therefore \ \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\dfrac{3}{2}\right]$

29. $\left.\begin{aligned} 3x+4y-11&=0 \\ 5y-6z&=-8 \\ 7z-8x-13&=0 \end{aligned}\right\}$

[*C. U.* 1877]

30. $\left.\begin{aligned} x+5y-4z&=5 \\ 3x-2y+2z&=14 \\ -10x+8y+z&=6 \end{aligned}\right\}$

[*C. U.* 1867]

31. $\left.\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y&=12-\tfrac{1}{6}z \\ \tfrac{1}{2}x+\tfrac{1}{3}z-\tfrac{1}{6}x&=8 \\ \tfrac{1}{2}x+\tfrac{1}{3}z&=10 \end{aligned}\right\}$

[*C. U.* 1868]

32. $\left.\begin{aligned} x-y-z&=-15 \\ y+x+2z&=40 \\ 4z-5x-6y&=-150 \end{aligned}\right\}$

[*C. U.* 1886]

33. $\left.\begin{aligned} 5x-3y+16z&=0 \\ 4x+7y&=6z \\ 3x-2y+19z&=9 \end{aligned}\right\}$

34. $\left.\begin{aligned} x+y&=axy \\ z+x&=bzx \\ z+y&=cyz \end{aligned}\right\}$

35. $\left.\begin{aligned} y-px+p^2z&=p^3 \\ y-qx+q^2z&=q^3 \\ y-mx+m^2z&=m^3 \end{aligned}\right\}$

PROBLEMS ON EQUATIONS

17. Many miscellaneous problems may be solved by Algebraical method. If unknown quantities involved in the problem be more than one, put x, y, z etc., for them. Observe the following examples.

Examples 6

Ex. 1. Twenty years ago a father was 4 times as old as his son and 4 years hence he will be twice as old as his son. What are their present ages ? [*C. U.* 1940]

Let the present ages of the father and the son be x and y years respectively. Now from the given conditions we have,

$$x-20=4(y-20)\cdots(1) \text{ and } x+4=2(y+4)\cdots(2).$$

From (1) we have $x-4y=-60\cdots(3)$ and from (2), $x-2y=4\cdots(4)$

From (3) − (4) we have $-2y=-64$, $\therefore\ y=32$;
and from (4) we have $x=4+64=68$.

$\therefore$ The present age of the father is 68 years and that of the son is 32 years.

Ex. 2. Add 1 to the numerator and the denominator of a certain fraction and it reduces to $\frac{4}{5}$; subtract 5 from each and it reduces to $\frac{1}{2}$; find the fraction. [*C. U.* 1916]

Let the fraction be $\dfrac{x}{y}$. Then from the given conditions we have

$$\frac{x+1}{y+1}=\frac{4}{5}\cdots(1) \text{ and } \frac{x-5}{y-5}=\frac{1}{2}\cdots(2).$$

From (1) we get $5x+5=4y+4$, or, $5x-4y=-1\cdots(3)$
and from (2) we have $2x-10=y-5$, or, $2x-y=5\cdots\cdots(5)$
Now, from (3) × 1 and (4) × 4 we have

$$\begin{array}{rl} & 5x-4y=-\ 1 \\ & 8x-4y=\ \ 20 \\ \hline \text{(subtracting)} & -3x\ \ =-21, \end{array}$$

$\therefore\ x=7$, and from (4), $14-y=5$,
or, $-y=-9$, $\therefore\ y=9$. $\therefore$ the reqd. fraction $=\frac{7}{9}$.

Ex. 3. Find the fraction which reduces to $\frac{1}{2}$ when 1 is added to its denominator, and $\frac{1}{3}$ when 2 is subtracted from its numerator. [*D. B.* 1932]

Let the fraction be $\frac{x}{y}$. From the given conditions we have,

$$\frac{x}{y+1}=\frac{1}{2}\cdots\cdots(1) \text{ and } \frac{x-2}{y}=\frac{1}{3}\cdots\cdots(2).$$

From (1) we have $2x-y=1\cdots(3)$, and from (2), $3x-y=6\cdots(4)$

Solving (3) and (4) we have $x=5$, $y=9$.

$\therefore$ the required fraction $=\frac{5}{9}$.

Ex. 4. *A* says to *B*, "I am twice as old as you were when I was as old you are". The sum of their present ages is 63 years. Find their ages. [*A. U.* '31]

Let the present ages of *A* and *B* be x and y years respectively.

Then from the given conditions we have $x+y=63 \cdots\cdots (1)$

and $x=2\{y-(x-y)\}^*$......(2).

From (2) we have $x=4y-2x$, or, $3x-4y=0\cdots\cdots(3)$.

Now, solving (1) and (3) we have $y=27$. $\therefore$ $x=63-27=36$.

$\therefore$ *A*'s age is 36 years and *B*'s age is 27 years.

*[*B*'s age is $(x-y)$ years less than *A*'s.

$\therefore$ when *A* was y years old, *B* was $y-(x-y)$ years old.]

Ex. 5. Nine chairs and 5 tables cost Rs. 90, while 5 chairs and 4 tables cost Rs. 61. Find the price of 6 chairs and 3 tables. [*P. U.* '30]

Let the prices of each chair and each table be x and y rupees respectively.

By the problem we have $9x+5y=90$......(1)

and $5x+4y=61$......(2)

Solving (1) and (2) we have $x=5$ and $y=9$.

$\therefore$ the price of 6 chairs and 3 tables $=$ Rs. $(5\times6+9\times3)=$ Rs. 57.

Ex. 6. The area of a floor is 192 sq. ft. Had each of the sides been 2 ft. longer, the area would have been increased by 60 sq. ft. Find the sides of the floor. [*C. U.* 1924]

Let the length and the width be x and y ft. respectively.

Then from the given conditions we have

$xy=192\cdots\cdots(1)$ and $(x+2)\times(y+2)=192+60\ldots(2)$.

$\therefore$ from (2) we have $xy+2x+2y=248$,

or, $192+2(x+y)=248$, or, $2(x+y)=56$, or, $x+y=28\ldots(3)$.

Now, $(x-y)^2=(x+y)^2-4xy=28^2-4\times192=16$.

$\therefore$ $x-y=4.\cdots(4)$. Solving (3) and (4) we have $x=16$, $y=12$.

$\therefore$ the required length $=16$ ft., and breadth $=12$ ft.

Ex. 7. The perimeter of a rectangular courtyard is 60 ft. If the length is increased by 3 ft. and the width be decreased by 3 ft., the area is decreased by 21 sq. ft. Find the dimensions of the courtyard. [*C. U.* 1927]

Let the length and the width be x and y feet respectively,

Then from the first condition $2(x+y)=60$, or, $x+y=30\ldots(1)$

From the second condition $(x+3)(y-3)=xy-21$,

or, $xy-3x+3y-9=xy-21$, or, $x-y=4\cdots(2)$.

Now, solving (1) and (2) we have $x=17$, $y=13$.

$\therefore$ the required length $=17$ ft. and the width $=13$ ft.

18. Problems relating to Digits

Ex. 1. A certain number consisting of two digits is equal to eight times the sum of its digits , if 45 be subtracted from the number, the digits interchange their places. Find the number. [*C. U.* 1919]

Let x be the digit in the units' place and y in the tens' place. Then the number is $10y+x$.

Now, from the given conditions we have

$10y+x=8(x+y)$...(1) and $10y+x-45=10x+y$...(2)

$\therefore$ from (1) we have $2y-7x=0$......(3)

and from (2) we have $9y-9x=45$, or, $y-x=5$...(4).

Solving (3) and (4) we have $x=2$ and $y=7$.

$\therefore$ the required number $=10\times7+2=72$.

[*N. B.* x and y denote the digits and not the number. The number is $10y+x$. If z were the digit in hundreds' place, the number would have been $100z+10y+x$.]

Ex. 2. The sum of the digits of a number less than 100 is 6; if the digits be reversed the resulting number will be less by 18 than the original number. Find the number. [*C. U.* '25]

As the number is less than 100, it evidently consists of 2 digits.

Let x and y be the digits in the units' place and the tens' place respectively. Then the number is $10y+x$.

Now, from the given conditions we have $x+y=6$$\cdots$(1)

and $10x+y=10y+x-18$...(2).

From (2) we have $9x-9y=-18$, or, $x-y=-2$$\cdot\cdot$ (3)

Now, solving (1) and (3) we have $x=2$ and $y=4$

$\therefore$ the required number $=10\times4+2=42$

✓**Ex. 3.** A number consists of two digits; the sum of the digits is 11, and if the left digit be increased by 2, it (the digit) will be equal to $\frac{1}{8}$th of the number. Find the number. [*C. U.* '36]

Let the number be $10y+x$.

Now from the given conditions we have $x+y=11$$\cdot\cdot$ (1)

and $y+2=\frac{1}{8}(10y+x)$$\cdots$(2)

From (2) we have $8y+16=10y+x$, or, $2y+x=16$$\cdots$(3).

Solving (1) and (3) we have $x=6$, $y=5$.

$\therefore$ the required number $=5\times10+6=56$.

✓**Ex. 4.** The sum of a number consisting of two digits and the number formed by interchanging the digits is 110 and the difference between the digits is 6. Find the number. [*A. U.* '28]

Let the number be $=10y+x$.

Then, $(10y+x)+(10x+y)=110 \cdots (1)$

and either $x-y=6$ or $y-x=6 \cdots (2)$.

From (1) we have $11x+11y=110$, or, $x+y=10 \cdots (3)$

Now, solving (2) and (3) we have $x=8$, $y=2$,

or, $x=2$, $y=8$.

$\therefore$ the required number is 28 or 82.

[The number is 28 when the digit in units' place is greater, and the number is 82 when this digit is less than the other digit.]

✓**Ex. 5.** One of the digits of a number is greater by 5 than the other. When the digits are inverted the number becomes $\frac{3}{8}$ths of the original number. Find the number. [*D. B.* '28]

Let x be the digit in the units' place and y in the tens' place.

Then the number $=10y+x$.

[*N. B.* Since the new number formed by inverting the digits is less than the original number, it is evident that the digit in the units' place of the original number is less than the digit in the tens' place. When the digit in the units' place is greater than that in the tens' place, the number formed by interchanging the digits would be greater than the original number.]

From the given conditions we have

$y-x=5 \ldots (1)$ and $10x+y=\frac{3}{8}(10y+x) \cdots (2)$.

From (2) we have, $80x+8y=30y+3x$, or, $77x-22y=0$,

or, $7x-2y=0 \ldots (3)$.

Now, solving (1) and (3) we have $x=2$, $y=7$.

$\therefore$ the required number $=7 \times 10+2=72$.

✓Ex. 6. There is a number of two digits whose difference is 2, and if it be diminished by $\frac{3}{2}$ times the sum of the digits, the digits will be inverted ; find it. [*C. U.* 1899]

Let x be the digit in the units' place and y in the tens' place.

Then the number $= 10y + x$.

Since the number formed by inverting the digits is less than the given number, x must be less than y.

$\therefore \; y - x = 2 \cdots (1)$, and $(10y + x) - \frac{3}{2}(x + y) = 10x + y \cdots (2)$.

From $(2) \times 2$ we get $20y + 2x - 3x - 3y = 20x + 2y$,

or, $15y - 21x = 0$, or, $5y - 7x = 0 \cdots\cdots (3)$.

Solving (1) and (3) we have $x = 5$, and $y = 7$.

$\therefore$ the required number $= 7 \times 10 + 5 = 75$.

Ex. 7. A number consists of three digits of which the middle one is 0 and the sum 8 ; the number formed by interchanging the digits is greater than the number itself by 198. Find the number. [*C. U.* 1922]

Let x be the digit in the units' place, 0 in the tens' place and y in the hundreds' place. Then the number $= 100y + x$.

From the given conditions we have $x + y = 8 \ldots\ldots (1)$

and $100x + y = 100y + x + 198 \cdots\cdots (2)$.

From (2) we have $99x - 99y = 198$, $x - y = 2 \cdots\cdots (3)$

Solving (1) and (3) we have $x = 5$, $y = 3$.

$\therefore$ the required number $= 3 \times 100 + 5 = 305$.

Ex. 8. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two ; and the number will be increased by 99 if the first and third digits be interchanged. Find the number. [*C. U.* 1923]

[*N. B.* Note that when we write the number 325, we first write 3, then 2 and then 5. Hence the first digit is 3 (and not 5).]

Let x, y, z be respectively the digits in the units', tens' and hundreds' places. Then the number $= 100z + 10y + x$.

Now, from the given conditions we have

$$x+y+z=10\cdots(1),\ x+z=y\cdots(2)$$

$$\text{and } 100x+10y+z=100z+10y+x+99\cdots(3).$$

From (1) and (2) we have $x+(x+z)+z=10$,

or, $2(x+z)=10$, or, $x+z=5\cdots(4)$.

From (3) we have $99x-99z=99$, or, $x-z=1\cdots(5)$.

Now solving (4) and (5) we have $x=3$, $z=2$, $\therefore$ $y=5$.

$\therefore$ the required number $=2\times100+5\times10+3=253$.

Ex. 9. A number consists of 3 digits, each greater by 2 than that which precedes it; if 16 be subtracted from the number, the remainder will be less than 20 times the sum of the digits by 10. Find the number.

Let x be the digit in the hundreds' place. Then from the given condition $x+2$ and $x+4$ will be the digits in the tens' and units' places respectively.

So the number $=100x+10(x+2)+(x+4)=111x+24$.

Now, from the second condition we have

$$111x+24-16=20(x+x+2+x+4)-10,$$

or, $111x+8=60x+110$, or, $51x=102$, $\therefore$ $x=2$.

$\therefore$ the required number $=111\times2+24=246$.

Ex. 10. A number consists of three consecutive digits. If the digits are reversed, the difference of the numbers is 33 times the greatest digit involved. Find the number. [*C. U.* 1939]

Let x be the digit in the hundreds' place. Then by the problem $x+1$ is the digit in tens' place and $x+2$ in units' place.

$\therefore$ the number $=100x+10(x+1)+(x+2)=111x+12$.

From the second condition we have

$$\{100(x+2)+10(x+1)+x\}-(111x+12)=33(x+2),$$

or, $111x+210-111x-12=33x+66$, or, $33x=132$. $\therefore$ $x=4$.

$\therefore$ the required number $=4\times111+12=456$.

19. Problems relating to the motion of river and boat

In still water, i. e., when there is no current, the distance traversed by a boat in 1 hour is equal to its speed per hour. If, however, there is a current or stream, (i) in going with the stream or current (i.e., going down-stream or down the river), the boat in 1 hour goes a distance equal to the sum of the rates of motion of the current and the boat, but (ii) while going against the current or up-stream or up the river, the distance traversed by the boat in 1 hour is the difference of their rates. Suppose the speed of the boat in still water is x miles per hour and that of the current is y miles per hour. Then the boat can go $x+y$ miles per hour down-stream and $x-y$ miles per hour up-stream.

Ex. 1. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours; it also goes upstream 40 miles and down-stream 55 miles in 13 hours. Find the rate of the stream and of the boat. [*D. B.* 1933]

Let x and y miles per hour be the rates of the boat and the stream respectively. Then the boat goes $x+y$ miles per hour down stream and $x-y$ miles per hour up-stream.

Now, from the given conditions $\dfrac{30}{x-y}+\dfrac{44}{x+y}=10 \ldots\ldots(1)$

$$\text{and } \frac{40}{x-y}+\frac{55}{x+y}=13 \cdots\cdots(2).$$

From $(1)\times 4$ and $(2)\times 3$ we have $\dfrac{120}{x-y}+\dfrac{176}{x+y}=40$

$$\text{and } \frac{120}{x-y}+\frac{165}{x+y}=39$$

$$(\text{ Subtracting }),\ \frac{11}{x+y}=1, \quad \therefore\ x+y=11\ldots(3)$$

Putting 11 for $x+y$ in (1) we have $x-y=5\ldots(4)$

Solving (3) and (4) we have $x=8$, $y=3$.

$\therefore$ The rate of the boat is 8 miles per hour and that of the stream is 3 miles per hour.

Ex. 2. A man rowing at the rate of 5 miles an hour in still water takes thrice as much time in going 40 miles up a river as in going 40 miles down; find the rate at which the river flows. [*C. U.* 1935]

Let x miles per hour be the rate of motion of the river or stream, then the boat goes $5-x$ miles up the river and $5+x$ miles down the river in 1 hour. $\therefore$ In going 40 miles up the river and down the river it takes $\frac{40}{5-x}$ and $\frac{40}{5+x}$ hours respectively.

$\therefore$ From the given condition we have $\frac{40}{5-x}=3\times\frac{40}{5+x}$

or, $\frac{1}{5-x}=\frac{3}{5+x}$, or, $5+x=15-3x$, or, $4x=10$,

$\therefore$ $x=2\frac{1}{2}$. $\therefore$ the river flows at $2\frac{1}{2}$ miles per hour.

Ex. 3. A person rowed down a river, a distance of 70 miles in 10 hours with the stream and rowed back again in 70 hours. Find the rate of the flow of the river per hour. [*C. U.* '41]

Let the rates of speed of the river and the boat be x and y miles per hour respectively. From the given conditions we have

$$10(x+y)=70\cdots(1) \text{ and } 70(y-x)=70\cdots(2)$$

From (1) we have $x+y=7\cdots(3)$, and from (2), $y-x=1\cdots(4)$.

Solving (3) and (4) we have $x=3$.

$\therefore$ the rate of flow of the river is 3 miles per hour.

20. Problems relating to Clocks

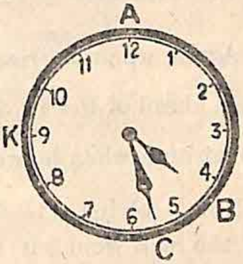
You know that the dial of a clock is divided into 60 small divisions. Each division is called the *minute division* or *minute space*. While the *minute-hand* passes over 60 minute divisions, the *hour-hand* passes over 5 minute divisions only.

You have already learnt easy problems relating to clocks. Some difficult problems are worked out here

Ex. 1. A man who went out between 4 and 5 P. M. and returned between 5 and 6 P. M. found that the hands of his watch had exactly changed places. When did he go out ? [*C. U.* '30, '51]

Suppose the man went out at x minutes past 4, and then the hour-hand was at B between 4 and 5 and the minute-hand was at C between 5 and 6. At x minutes past 4, the minute-hand is x minute divisions ahead of the 12 o'clock mark and the hour-hand $\frac{x}{12}$ minute divisions ahead of the 4 o'clock mark.

From the figure we find,



arc $AC = x$ minute spaces

arc $AB = 20 + \frac{x}{12}$ „ „

arc $CKA = 60 - x$ „ „

arc $BC = x - \left(20 + \frac{x}{12}\right)$ „ „

Now during the period the man stayed outside the minute-hand passed over the arc $CKAB$, i.e. $\left(60 - x + 20 + \frac{x}{12}\right)$ minute divisions, and the hour-hand passed over the arc BC, *i.e.*, $x - \left(20 + \frac{x}{12}\right)$ minute divisions.

As the minute-hand moves 12 times as fast as the hour-hand, we have $60 - x + 20 + \frac{x}{12} = 12\left\{x - \left(20 + \frac{x}{12}\right)\right\}$,

or, $80 - \frac{11x}{12} = 11x - 240$, or, $\frac{143}{12}x = 320$, $\therefore$ $x = \frac{3840}{143} = 26\frac{122}{143}$

Hence, the man went out at $26\frac{122}{143}$ minutes past 4 P. M.

Ex. 2. A man who went out between 3 P. M. and 4 P. M. and returned between 4 P. M. and 5 P. M. found that the hands of the clock had exactly changed places. When did he go out ? When did he return and how long did he stay outside ? [*Cf. C. U.* 1942]

[**Second Method**] Suppose the man went out at x minutes past 3 and returned at y minutes past 4. While the minute-hand moves over x and y minute divisions, the hour-hand passes over $\frac{x}{12}$ and $\frac{y}{12}$ minute divisions respectively.

$\therefore$ when he went out, the hour-hand was $15+\frac{x}{12}$ minute divisions ahead of the 12 o'clock mark, and the minute-hand was at this position when he returned at y minutes past 4.

$\therefore$ evidently $y=15+\frac{x}{12}\cdots(1)$

Again, when he returned, the hour-hand was $20+\frac{y}{12}$ minute spaces ahead of the 12 o'clock mark and the minute-hand was at this position when he went out, $\therefore$ $x=20+\frac{y}{12}\ldots(2)$.

Now, solving (1) and (2) we have $x=21\frac{57}{143}$ and $y=16\frac{112}{143}$. $\therefore$ the man went out at $21\frac{57}{143}$ minutes past 3 P. M,. returned at $16\frac{112}{143}$ minutes past 4 P. M., and the time he stayed outside is equal to the difference of the two times, i.e., $55\frac{5}{13}$ minutes.

[Miscellaneous Problems]

Ex. 1. A person bought an article and sold it at a profit of 6%. Had he bought it at 4 p.c. less and sold it at Rs. 2. 6 as. more, his profit would have been 12%. For how much did he buy it ?

[*C. U.* 1944]

Let the cost price be x rupees. As the article was sold at a profit of 6%, the sale price was $\frac{106}{100}x$ rupees. Had it been bought at 4% less and sold at a profit of 12%, the cost price would have been $\frac{96x}{100}$ rupees and the sale price $\frac{112}{100}\times\frac{96x}{100}$ rupees.

$\because$ the second sale price is Rs. 2.6 as. or $\frac{19}{8}$ rupees more than the first sale price,

$\therefore$ $\frac{112\times96}{100\times100}x=\frac{106}{100}x+\frac{19}{8}$, or, $\frac{672x}{625}-\frac{53x}{50}=\frac{19}{8}$, or, $\frac{19x}{1250}=\frac{19}{8}$,

$\therefore$ $x=\frac{19}{8}\times\frac{1250}{19}=156\frac{1}{4}$. $\therefore$ the reqd. cost price = Rs. 156. 4 as.

Ex. 2. Divide the nmmber **77** into three parts such that the sum of the first and second multiplied by 3, the sum of the second and third diminished by 3, and the sum of the first and third increased by 3 may be all equal. [*A. U.* '33]

Let x, y, z be the three parts.

Then by the problem we have $x+y+z=77$......(1)

and $3(x+y)=y+z-3=x+z+3$......(2)

Now, $3x+3y=y+z-3$, or, $3x+2y-z=-3$...(3)

Again, $3x+3y=x+z+3$, or, $2x+3y-z=3$......(4)

$$\left.\begin{aligned}\therefore\ \text{from } (3)+(4) \text{ we have, } & 5x+5y-2z=0\\ \text{and from } (1)\times 5 \text{ we have, } & 5x+5y+5z=385\end{aligned}\right\}$$

(Subtracting) $-7z=-385$, $\therefore\ z=55$.

$\therefore$ from (1) we have $x+y=77-55=22$......(6)

and from the last part of (2) we have $y-x=6$...(7)

Solving (6) and (7) we have $x=8$, $y=14$.

$\therefore$ the required parts are 8, 14 and 55.

Ex. 3. Two mixtures contain wine and water in the ratio of $2:3$ and $5:4$ respectively; in what ratio must the two mixtures be mixed together so that the resulting mixture may contain equal quantities of wine and water ?

Let x quantity of the first mixture and y quantity of the second be mixed together.

Now, of the x quantity of the first mixture, $\frac{2}{5}x$ is wine and $\frac{3}{5}x$ is water and of the y quantity of the second mixture, $\frac{5}{9}y$ is wine and $\frac{4}{9}y$ is water.

$\therefore$ in $(x+y)$ quantity of the resulting mixture

$\frac{2}{5}x+\frac{5}{9}y$ is wine and $\frac{3}{5}x+\frac{4}{9}y$ is water.

$\therefore\ \frac{2}{5}x+\frac{5}{9}y=\frac{3}{5}x+\frac{4}{9}y$, or, $\frac{5}{9}y-\frac{4}{9}y=\frac{3}{5}x-\frac{2}{5}x$,

or, $\dfrac{y}{9}=\dfrac{x}{5}$, $\therefore\ \dfrac{x}{y}=\dfrac{5}{9}$. $\therefore$ the required ratio is $5:9$.

[*N. B.* If in the resulting mixture wine and water be in a given ratio, say $3:4$, then put $\dfrac{\frac{2}{5}x+\frac{5}{9}y}{\frac{3}{5}x+\frac{4}{9}y}=\dfrac{3}{4}$ and find $x:y$ from it.]

✓**Ex. 4.** A traveller walks a certain distance; had he gone half a mile an hour faster, he would have walked it in $\frac{4}{5}$ths of the time; had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

Let the whole distance be x miles and suppose the man walked y miles per hour. Then he walked x miles in $\frac{x}{y}$ hours.

Then from the given conditions we have

$$\frac{x}{y+\frac{1}{2}}=\frac{4}{5}\times\frac{x}{y}\cdots\cdots(1) \quad\text{and}\quad \frac{x}{y-\frac{1}{2}}=\frac{x}{y}+2\tfrac{1}{2}\cdots\cdots(2).$$

Now, from (1) we have $\frac{1}{y+\frac{1}{2}}=\frac{4}{5y}$, or, $5y=4y+2$, $\therefore\ y=2$.

$\therefore$ from (2) we have $\frac{x}{2-\frac{1}{2}}=\frac{x}{2}+2\frac{1}{2}$, or, $\frac{2x}{3}=\frac{x}{2}+\frac{5}{2}$,

or, $4x=3x+15$, $\therefore\ x=15$.

$\therefore$ the required distance $=15$ miles.

Ex. 5. If a sum of money be lent at 5%, the interest for a certain time exceeds the loan by Rs. 80; but if it be lent at 3% for a fourth of the time, the loan exceeds its interest by Rs. 328. Find the sum lent.

Suppose the sum of x rupees was lent for y years.

The interest on Rs. x at 5% for 1 year $=\frac{5x}{100}=\frac{x}{20}$

$\therefore$ " " " " y years $=\frac{xy}{20}$.

$\therefore\ \frac{xy}{20}=x+80\ldots\ldots(1)$

Again, the interest on Rs. x at 3% for $\frac{y}{4}$ yrs. $=\frac{3x}{100}\times\frac{y}{4}=\frac{3xy}{400}$

$\therefore\ x=\frac{3xy}{400}+328\cdots(2)$

Now, from (1) we have $xy=20(x+80)\ldots\ldots(3)$

and from (2), $\frac{3xy}{400}=x-328$, or, $xy=\frac{400(x-328)}{3}\ldots(4)$

$\therefore\ \frac{400'x-328)}{3}=20(x+80)$, or, $\frac{20(x-328)}{3}=x+80$,

or, $20x-6560=3x+240$, or. $17x=6800$. $\therefore\ x=400$.

$\therefore$ the sum of Rs. 400 was lent.

Ex. 6. If the larger of two numbers is divided by the smaller, the quotient and remainder are each 3. The result will be the same, if 10 times the smaller is divided by the larger. Find the numbers.

Let x and y be the numbers of which $x > y$.

$\because$ the quotient and the remainder obtained by dividing x by y are each 3, $\therefore$ $x = 3y + 3 \cdots\cdots(1)$

Similarly from the second condition we have $10y = 3x + 3 \ldots(2)$.

Putting $3y + 3$ for x in (2) we have $10y = 3(3y + 3) + 3$,

or, $10y = 9y + 12$, $\therefore$ $y = 12$.

$\therefore$ From (1) we have $x = 3 \times 12 + 3 = 39$.

$\therefore$ the required numbers are 39 and 12.

Ex. 7. At a certain election there were two rival candidates, and their supporters were conveyed to the polling booths in carriages capable of accommodating 10 and 15 voters respectively. If the voters, 1050 in all, just filled 85 carriages, find by what majority the election was won.

Let x be the number of voters for the first candidate and y for the second.

Then from the given conditions we have $x + y = 1050 \cdots\cdots(1)$.

Again, $\because$ x voters were carried in $\frac{x}{10}$ carriages and y voters in $\frac{y}{15}$ carriages, $\therefore$ $\frac{x}{10} + \frac{y}{15} = 85 \cdots\cdots(2)$

Solving (1) and (2) we have $x = 450$ and $y = 600$.

$\therefore$ the second candidate won by $(600 - 450)$ or 150 votes.

Ex. 8. A train 88 yds. long passed another train 110 yds. long travelling in the same direction on a parallel line of rails in 27 seconds, but had the slower train been running half as fast again, it would have been passed in twice that time. Find the rates at which the trains were travelling.

Let x and y miles per hour be the rates of speed of the two trains, of which the second one is slower.

In the first case the time$=27$ sec.$=\frac{3}{400}$ hr.

In the second case the time$=\frac{3}{200}$ hr. The trains run in the same direction, $\therefore$ their relative motion is $(x-y)$ miles per hour.

The total length of the trains$=(110+88)$ yds.$=198$ yards

$$=\tfrac{198}{1760}\text{ mi.}=\tfrac{9}{80}\text{ mile.}$$

The time taken to run $\frac{9}{80}$ mile at $(x-y)$ miles per hour is $\dfrac{9}{80(x-y)}$ hr.

$$\therefore\quad \frac{9}{80(x-y)}=\frac{3}{400},\ \text{ or, }\quad x-y=15\cdots\cdots(1)$$

In the second case, the speed of the second train is $\frac{3}{2}y$ miles per hour.

$\therefore$ the relative speed of the two trains is $(x-\frac{3}{2}y)$ miles per hr.

$$\therefore\quad \frac{9}{80(x-\frac{3}{2}y)}=\frac{3}{200},\qquad \text{or,}\quad 2x-3y=15\cdots\cdots(2)$$

Now, multiplying (1) by 2 we have

$$2x-2y=30$$

and (2) is $2x-3y=15$

$\therefore$ (Subtracting) $y=15$. $\quad\therefore$ from (1), we have $x=30$.

$\therefore$ the rates of the two trains are 30 miles and 15 miles per hour respectively.

Ex. 9. A certain number of men paid a bill. If there had been 10 more, each would have paid 1*s.* less ; but if there had been 5 fewer, each would have paid 1*s.* more. Find the number of men and what each had to pay.

Suppose x was the number of men and each had to pay y shillings. Then the bill was for xy shillings. From the given conditions we have $\dfrac{xy}{x+10}=y-1\cdots(1)$ and $\dfrac{xy}{x-5}=y+1\cdots\cdots(2)$.

From (1) we have $xy=xy-x+10y-10$, or, $x-10y=-10$...(3)
From (2) „ „ $xy=xy+x-5y-5$, $\underline{\text{or, } x-5y=\ \ 5}$...(4)
∴ $-5y=-15$, ∴ $y=3$.

Now, from (4) we have $x-5\times3=5$, ∴ $x=20$.

∴ The required number of men $=20$, and each had to pay 3*s*.

Ex. 10. In a race of 1 mile, *A* gives *B* a start of 88 yds. and beats him by a minute and a half; but *A* is beaten by 44 yds., if he gives *B* a start of 2 minutes 12 seconds. In what time can *A* and *B* run a mile?

Suppose *A* takes x minutes and *B* takes y minutes to run 1 mile. Then *A* goes $\frac{1760}{x}$ yds. and *B* goes $\frac{1760}{y}$ yds. per minute.

In the first case, *B* is 88 yds. ahead of *A* but is beaten by 88 yds. ∵ *A* goes 1 mile in x minutes,

∴ *B* takes $(x+1\frac{1}{2})$ minutes to go $(1760-88)$ or 1672 yds.

$$\therefore\ 1672\div\frac{1760}{y}=x+1\tfrac{1}{2},\ \text{ or, }\ \frac{1672y}{1760}=\frac{2x+3}{2},$$

$$\text{or, }\ 20x-19y=-30\cdots\cdots(1).$$

In the second case when *B* reaches the destination, *A* is 44 yds. behind him, *i.e.*, *A* has gone $(1760-44)$ or 1716 yds. ∵ *B* has started $2\frac{1}{5}$ minutes before *A*, ∴ he has gone the remaining distance in $(y-2\frac{1}{5})$ minutes.

∴ *A* has gone 1716 yds. in $(y-2\frac{1}{5})$ minutes.

$$\therefore\ 1716\div\frac{1760}{x}=y-2\tfrac{1}{5},\ \text{ or, }\ \frac{1716x}{1760}=\frac{5y-11}{5},$$

or, $39x-40y=-88\cdots\cdots(2)$. Multiplying (1) by 40 and (2) by 19 we have $800x-760y=-1200$

$$\underline{\text{and }\ 741x-760y=-1672}$$

∴ (subtracting) $59x=\ 472$, ∴ $x=8$.

∴ From (1) we have $y=10$.

∴ *A* and *B* can run 1 mile in 8 and 10 minutes respectively.

Exercise 5

1. Ten years ago, a father was seven times as old as his son; two years hence twice his age will be equal to 5 times his son's; what are their present ages? [*C. U.* 1920]

2. A motorist does a journey of 80 miles in 6 hours. During the first part of the journey he travels at the rate of 10 miles and during the latter part at 18 miles an hour. How far does he travel at each rate? [*C. U.* '18, '29]

3. A number consists of two digits. The sum of the digits falls short of the number by 54, if the digits be reversed the number exceeds the old number by 27; find the number. [*P. U.* '35]

4. The product of two numbers is 18225 and the quotient when the larger number is divided by the smaller is 81. Find the numbers. [*C. U.* 1945]

5. Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from the denominator. [*C. U.* 1928]

6. A number consists of two digits. The digit in the tens' place is 3 times the digit in the units' place. If 54 is subtracted from the number, the digits are inverted. Find the number. [*C. U.* 1943]

7. A certain number between 10 and 100 is 8 times the sum of its digits and if 45 be subtracted from it, the digits will be reversed. Find the number. [*C. U.* 1919]

8. Reverse the digits of a number and it will become $\frac{5}{6}$ths of what it was before; and also the difference between the two digits is 1. Find the number. [*C. U.* 1883, '49 Sup.]

9. A man who went out for an evening walk between 5 and 6, returned between 6 and 7 and found that the hands of the clock had exactly changed places. When did he go out? [*C. U.* '44]

10. A boy spends his money in oranges. If he had received 4 more for his money, they would have averaged a half-penny each less; if three less, a half-penny each more. How much did he spend? [*P. U.* 1921, Addl.]

11. A man rowing at the rate of 4 miles an hour takes thrice as much time in going 30 miles up a river as in going 30 miles down; find the rate at which the river flows.

12 A mixture contains wine and water in the ratio of 5 : 3 and another in the ratio of 4 : 5. In what ratio must the two kinds of mixture be mixed to give a mixture of wine and water in the ratio of 31 : 29?

13. How much gold, at Rs. 20 a tola, must be mixed with 14 tolas of Rs. 15 a tola, so that the mixture may be worth Rs. 18 a tola? [*P. U.* 1921]

14. If a cyclist had gone 2 miles an hour faster, he would have taken 1 hour 40 minutes less to ride 100 miles. What time did he take? [*P. U.* 1925]

15. A number consists of two digits, the digit in the units' place being four times that in the tens' place. If the digits be inverted, the new number increased by 2 equals three times the old number. Find the number. [*C. U.* 1901]

16. Two men, 40 miles apart, walking in opposite directions, meet in $6\frac{2}{3}$ hours; but if one of them had doubled his pace, they would have met in $\frac{3}{4}$ths. of the time. Find their respective speeds. [*P. U.* 1931]

17. A man rows 30 miles down a river in 6 hours and returns in 10 hours. Find the rate at which the man rows, and also the rate at which the river flows. [*P. U.* 1933]

18. A number consists of two digits; the digit in the tens' place is twice the digit in the units' place; if 36 be subtracted from the number, the digits are inverted; find the number. [*C. U.* 1946]

19. The number of months in the age of a man on his birth-day in the year 1875 was exactly half of the number denoting the year in which he was born. In what year was he born? [*A. U.*]

20. A boy buys a certain number of oranges at 3 for 2*d.*, and one-third of that at 2 for 1*d.*; at what price must he sell them to get 20% profit? If his profit be 5*s.* 4*d.*, find tho number bought. [*D. B.* 1936]

21. One customer buys 14℔. of tea and 10℔. of coffee for £2. 3*s.*, and another buys 11℔. of tea and 15℔ of coffee for £2. 4*s.* 6*d.* Find the price of tea and coffee per ℔. [*D. B.* '40]

22. A number consists of three digits each less by unity than that which follows it and if 3 be subtracted from the number, the remainder will be 20 times the sum of the digits. Find the number. [*G. U.* '48]

23. *P* and *Q* start at the same time from Howrah and Madhupur and proceed towards each other at the rate of 20 and 30 miles per hour respectively. They meet when *Q* has proceeded 36 miles farther than *P*. Find the distance between Howrah and Madhupur. [*C. U.* '49]

24. If the numerator of a certain fraction is doubled and its denominator increased by 1, its value becomos $\frac{1}{2}$; but if its denominator is doubled and its numerator increased by 1, its value becomes $\frac{1}{6}$. Find the fraction. [*E. B. S. B.* '55]

25. A man spent 15*s.* 2*d.* in buying oranges at the rate of 3 for 2 pence and apples at five pence a dozen; if he had bought 5 times as many oranges and $\frac{1}{4}$ of the number of apples he would have spent £2. 4*s.* 2*d.* How many of each did he buy?

26. Find the distance between two towns when by increasing the speed 7 miles per hour a train can perform the journey in 1 hour less, and by reducing the speed 5 miles an hour can perform the journey in 1 hour more.

27. The middle digit of a number between 100 and 1000 is zero and the sum of the other digits is 16. If the digits be reversed, the number so formed exceeds the original number by 198; find it.

28. The three sides of a triangle are $x+5$, $4x-y$ and $y+2$ inches in length. If the triangle is equilateral, find its perimeter.

29. If the sum of the digits of a number is divisible by 3, show that the number is divisible by 3.

30. If $\frac{1}{3}$ be added to the numerator of a certain fraction, the fraction is increased by $\frac{1}{15}$, and if $\frac{1}{4}$ be taken from the denominator the fraction becomes $\frac{8}{19}$. Find the fraction.

31. The incomes of two men are in the ratio of 5 : 3 and their expenditures are as 9 : 5. Each saves Rs. 30 a year. Find their incomes.

32. A man walks a certain distance. Had he gone half a mile faster, he would have walked it in $\frac{6}{7}$ths of the time; and had he gone $\frac{2}{3}$ mile an hour slower, he would have taken 2 hours longer. Find the distance and the rate at which he walked.

33. If a sum of money be lent out at 8%, the interest for a certain time exceeds the loan by Rs. 60; but if it be lent at 6% for half the period, the loan exceeds the interest by Rs. 290. Find the sum.

34. If the larger of two numbers is divided by the smaller the quotient and the remainder are each 4. The result will be the same, if 20 times the smaller is divided by the larger. Find the numbers.

✓**35.** *A* can run 50 yds. whilst *B* runs 45 yds.; if *B* has 5 minutes' start in a race, what time will *A* take to get level with *B*?

36. A train 44 yards long passed another train 66 yards long travelling in the same direction in $22\frac{1}{2}$ seconds. Had the slower train been running half as fast again, it would have been passed in 45 seconds. Find the rates at which the trains were travelling.

✓37. In a race of 1320 yards A gives B a strat of 176 yards and beats him by 15 seconds, but A is beaten by 110 yards, if he gives B a start of one minute. In what time can A and B run a mile ?

✓38. In a mile race A can win by half a minute against B, and by 88 yards against C ; also B can win by 20 seconds against C. Find the time required by C in running a mile. [S. F. Addl. '60]

39. In a race of 2640 yds., A can beat B by 10 yds. and C by 30 seconds. In the same race B can beat C by 24 secs. How much time does C take to run a mile ? [S. F. Addl. '57]

40. Show that if a number formed of two digits is four times the sum of the digits, the number formed by interchanging the digits is seven times the sum of the digits. [S. F. '56]

41. A starts from a place P to go to a place Q ; at the same time B starts from Q for P. After meeting they arrive at their destinations in 2 and 3 hours respectively. Show that the ratio of their speeds (supposed to be uniform) is $\sqrt{3} : \sqrt{2}$. [S. F. Addl. '61]

42. The sum of the digits of a number of two digits is 10. If 18 be subtracted from the number, the two digits of the remainder are equal. Find the number.

THEORY OF INDICES

21. You know that a^3 stands for the product of 3 factors each equal to a. Thus $a^3 = a \times a \times a$. Similarly b^m stands for the product of m factors each equal to b.

Thus $b^m = b \times b \times b$...to m factors.

Here 3 and m indicate the number of equal factors.

Power : The product of several factors each equal to the same number is called a **power** of that number.

Thus a^3, b^m are powers of a and b.

Index : The small figure that indicates the number of equal factors is called the **index** or **exponent** of the power.

Thus in a^3, b^m the indices are 3, m respectively.

22. Laws relating to Indices

Law 1. To prove that $a^m \times a^n = a^{m+n}$ where m and n are positive integers. [*C. U.* '10, '30, ; *D. B.* '30, *Pat. U.* '21]

By definition, $a^m = a \times a \times a \times \cdots\cdots$ to m factors.

and $a^n = a \times a \times a \times \cdots\cdots$ to n factors ;

$$\therefore a^m \times a^n = (a \times a \times a \times \cdots \text{to } m \text{ factors}) \times (a \times a \times a \times \cdots \text{to } n \text{ factors})$$
$$= a \times a \times a \times \ldots \text{to } (m+n) \text{ factors} = a^{m+n}.$$

N. B. This result is known as the *Fundamental Index Law.*

If p is also a positive integer, then $a^m \times a^n \times a^p = a^{m+n+p}$: and so on for any number of factors.

Law 2. To prove that $a^m \div a^n = a^{m-n}$, where m and n are positive integers and m is greater than n.

By definition, $a^m \div a^n = \dfrac{a \times a \times a \times \ldots\ldots \text{to } m \text{ factors}}{a \times a \times a \times \ldots\ldots \text{to } n \text{ factors}}$.

Now, since m is greater than n, all the n factors of the denominator cancel with n factors in the numerator, leaving $(m-n)$ factors there.

$\therefore$ we have $a^m \div a^n = a \times a \times a \times \ldots\ldots$ to $(m-n)$ factors

$$= a^{m-n}.$$

[*N. B.* If, however, $n > m$, we have $a^m \div a^n = \dfrac{1}{a^{n-m}}$].

We have derived the above results on the supposition that the indices are all *positive integers*. But the indices may be *negative* or *fractional*.

(*a*) If the index m be a negative integer (say, -4), then a^m has no intelligible meaning, for $a^m = a^{-4} = a \times a \times a \ldots\ldots$ to (-4) factors by definition. But it is meaningless, as we cannot find the product of -4 number of equal factors.

(*b*) Let the index $m = \frac{4}{5}$.

Then by definition $a^m = a^{\frac{4}{5}} = a \times a \times a \times \cdots\cdots$ to $\frac{4}{5}$ number of factors, which also is meaningless.

(*c*) Similary a^m can have no intelligible meaning, if $m = -\frac{4}{5}$ or 0.

Now, we have to see if the laws of indices in the above cases can be proved, so that all indices may be governed by the same laws.

Corollary (*i*) To find the meaning or value of $\mathbf{a}^\circ$.

By the Index Law $a^m \times a^n = a^{m+n}$. If we replace the index m by 0, we have $a^0 \times a^n = a^{0+n} = a^n$. $\therefore$ $a^0 = \dfrac{a^n}{a^n} = 1$.

Thus any number or expression with the index 0 is equal to 1.

(*ii*) To determine a meaning for $\mathbf{a}^{-m}$.

$\because$ $a^m \times a^n = a^{m+n}$ by the Index Law,

$\therefore$ putting $-m$ for n we have $a^m \times a^{-m} = a^{m+(-m)}$

$= a^{m-m} = a^0 = 1$. $\therefore$ $a^{-m} = \dfrac{1}{a^m}$. Thus a^{-m} is the reciprocal of a^m.

So any number with a negative index is equal to the reciprocal of the number with same index of positive sign.

$$\text{Thus } a^{-2}=\frac{1}{a^{2}},\ \frac{1}{a^{-3}}=a^{3},\ \frac{a^{-m}}{b^{-n}}=\frac{b^{n}}{a^{m}}.$$

Law 3. To prove that $(a^{m})^{n}=a^{mn}$. [*C.* U. ; *D. B.* ; *P.* U.]

(*i*) If n is a positive integer, we have

$$(a^{m})^{n}=a^{m}\times a^{m}\times a^{m}\times\cdots\cdots\text{to } n \text{ factors}$$
$$=a^{m+m+m+\ldots\ldots\text{to } n \text{ terms}}=a^{mn}.$$

Cor. (*a*). $(a^{mn})^{p}=a^{mnp}$, $(a^{m})^{np}=a^{mnp}$.

(*b*) The meaning of $a^{\frac{m}{n}}$.

$$\because\ \left(a^{\frac{m}{n}}\right)^{n}=a^{\frac{m}{n}\times n}=a^{m},\ \therefore\ a^{\frac{m}{n}} \text{ is the } n\text{th root of } a^{m}.$$

Again, $\because\ a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^{m}$, $\therefore\ a^{\frac{m}{n}}$ is also the mth power of $a^{\frac{1}{n}}$.

[N. B. If the index of a number be a fraction, the numerator of the fraction is taken as the power and the denominator as the root of the number. Thus $a^{\frac{m}{n}}$ is the nth root of the mth power of a, or it is the mth power of the nth root of a.

Thus, $a^{\frac{1}{2}}=\sqrt{a}$; $a^{\frac{2}{5}}=(\sqrt[5]{a})^{2}=(a^{\frac{1}{5}})^{2}$; $\sqrt[3]{a^{2}}=a^{\frac{2}{3}}$; $\sqrt[p]{a^{q}}=a^{\frac{q}{p}}$.]

(*ii*) If in Law 3, n is a fraction, suppose $n=\frac{p}{q}$ where p and q are each a positive integer.

$$\text{Now, }\ (a^{m})^{n}=(a^{m})^{\frac{p}{q}}=\sqrt[q]{(a^{m})^{p}}=\sqrt[q]{a^{mp}}=a^{\frac{mp}{q}}=a^{mn}.$$

(*iii*) In Law 3, if n is a negative quantity,
let $n=-p$, where p is a positive integer.

$$\text{Now, }\ (a^{m})^{n}=(a^{m})^{-p}=\frac{1}{(a^{m})^{p}}=a^{-mp}=a^{m(-p)}=a^{mn}.$$

Thus, $(a^{m})^{n}=a^{mn}$, whatever may be the values of m and n.

(*iv*) What is the meaning of $a^{-\frac{m}{n}}$, where a is a positive quantity and m and n are both positive integers ?

$\because \; a^{-\frac{m}{n}}=\dfrac{1}{a^{\frac{m}{n}}}=\dfrac{1}{\sqrt[n]{a^m}}$, $\therefore \; a^{-\frac{m}{n}}$ means the reciprocal of the nth root of a^m.

[N. B. We do not consider the meaning of $a^{\frac{m}{n}}$ or $a^{-\frac{m}{n}}$ where a is a negative quantity, as it is beyond our scope.]

Law 4. To prove that $(ab)^m=a^mb^m$. [*C.* U. ; *D. B.*]

$(ab)^m=ab\times ab\times ab\times\cdots\cdots$ to m factors

$=(a\times a\times a\times\cdots\cdots$ to m factors$)\times(b\times b\times b\times\cdots\cdots$ to m factors$)$

$=a^m\times b^m=a^mb^m.$

Law 5. $\left(\dfrac{a}{b}\right)^m=\dfrac{a\times a\times a\times\cdots \text{ to } m \text{ factors}}{b\times b\times b\times\cdots \text{to } m \text{ factors}}=\dfrac{a^m}{b^m}.$

N. B. In the above laws we have discussed the meanings of a^m in the following cases :—

(*i*) Where a is any positive or negative quantity excepting 0 and m is a positive integer ;

(*ii*) Where a is any positive integer other than zero and m is a positive fraction ;

(*iii*) Where a is any positive or negative integer other than zero and m is a negative integer ;

(*iv*) Where a is a positive integer other than zero and m is a negative fraction.

Law 6. To prove that $a^mb^m=(ab)^m$, where m is a positive fraction.

Let $m=\dfrac{p}{q}$ (p and q being both positive integers).

Now, $(a^m b^m) = \left(a^{\frac{p}{q}}.b^{\frac{p}{q}}\right)$,

$\therefore \quad \left(a^m b^m\right)^q = \left(a^{\frac{p}{q}}\cdot b^{\frac{p}{q}}\right)^q = \left(a^{\frac{p}{q}}\right)^q \times \left(b^{\frac{p}{q}}\right)^q$ [Vide Law 4]

$= a^p \times b^p$ [Vide Law 3]

$= (ab)^p. \qquad \therefore \quad a^m b^m = (ab)^{\frac{p}{q}} = (ab)^m.$

Law 7. If m is a negative quantity, shew that $a^m b^m = (ab)^m$.

Let $m = -p$ (p being a positive quantity).

Now, $a^m b^m = a^{-p} b^{-p} = \dfrac{1}{a^p} \times \dfrac{1}{b^p} = \dfrac{1}{(ab)^p} = (ab)^{-p} = (ab)^m.$

INSTITUTE OF EDUCATION FOR
Dept. of Extension
SERVICE.
CALCUTTA-27

Examples 7

1. Find the product of $a^{\frac{1}{2}}\cdot a^{\frac{1}{4}}\cdot a$.

The product $= a^{\frac{1}{2}}.a^{\frac{1}{4}}.a = a^{\frac{1}{2}+\frac{1}{4}+1} = a^{\frac{7}{4}}$.

2. Find the value of $a^{\frac{1}{3}}.a^{\frac{1}{2}}.a^{-\frac{1}{6}} \div a^{\frac{2}{3}}$.

The required value $= a^{\frac{1}{2}+\frac{1}{3}-\frac{1}{6}-\frac{2}{3}} = a^0 = 1$.

3. Simplify $\left(x^2 y\right)^{\frac{1}{4}} \times \left(x^3\right)^{\frac{1}{2}} \times \left(x^{-4} y^{-1}\right)^{\frac{1}{2}}$.

The given exp. $= x^{2\times\frac{1}{4}}.y^{1\times\frac{1}{4}}.x^{3\times\frac{1}{2}}.x^{-4\times\frac{1}{2}}.y^{-1\times\frac{1}{2}}$

$= x^{\frac{1}{2}}.y^{\frac{1}{4}}.x^{\frac{3}{2}}.x^{-2}.y^{-\frac{1}{2}} = x^{\frac{1}{2}+\frac{3}{2}-2}.y^{\frac{1}{4}-\frac{1}{2}} = x^0 y^{-\frac{1}{4}}$

$= 1 \times y^{-\frac{1}{4}} = \dfrac{1}{y^{\frac{1}{4}}} = \dfrac{1}{\sqrt[4]{y}}.$

4. Find the value of $\sqrt[3]{27^2}$.

$$\sqrt[3]{27^2}=\left(27\right)^{\frac{2}{3}}=\left(3^3\right)^{\frac{2}{3}}=3^{3\times\frac{2}{3}}=3^2=9.$$

5. Find the value of $32^{-\frac{4}{5}}$.

$$32^{-\frac{4}{5}}=\frac{1}{32^{\frac{4}{5}}}=\frac{1}{\left(2^5\right)^{\frac{4}{5}}}=\frac{1}{2^{5\times\frac{4}{5}}}=\frac{1}{2^4}=\frac{1}{16}.$$

6. $$4a^{-\frac{3}{4}}\div 3b^{-\frac{2}{3}}=\frac{4a^{-\frac{3}{4}}}{3b^{-\frac{2}{3}}}=\frac{4\times\dfrac{1}{a^{\frac{3}{4}}}}{3\times\dfrac{1}{b^{\frac{2}{3}}}}=\frac{4}{3}\times\frac{b^{\frac{2}{3}}}{a^{\frac{3}{4}}}=\frac{4\sqrt[3]{b^2}}{3\sqrt[4]{a^3}}.$$

7. Multiply

$a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$ by $a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.

$$\begin{array}{llll}
 & a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1} & & \\
 & \underline{a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}} & & \\
 & a^{-2}+a^{-\frac{3}{2}}b^{-\frac{1}{2}}+a^{-1}b^{-1} & & \\
 & \quad -a^{-\frac{3}{2}}b^{-\frac{1}{2}}-a^{-1}b^{-1}-a^{-\frac{1}{2}}b^{-\frac{3}{2}} & & \\
 & \qquad\qquad +a^{-1}b^{-1}+a^{-\frac{1}{2}}b^{-\frac{3}{2}}+b^{-2} & & \\
\hline
\therefore \text{ the product} = & a^{-2}\qquad\qquad +a^{-1}b^{-1}\qquad\qquad +b^{-2} & &
\end{array}$$

Otherwise : $\left(a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}\right)\left(a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}\right)$

$$=\left\{\left(a^{-1}+b^{-1}\right)+a^{-\frac{1}{2}}b^{-\frac{1}{2}}\right\}\left\{\left(a^{-1}+b^{-1}\right)-a^{-\frac{1}{2}}b^{-\frac{1}{2}}\right\}$$

$$=\left(a^{-1}+b^{-1}\right)^2-\left(a^{-\frac{1}{2}}b^{-\frac{1}{2}}\right)^2=a^{-2}+b^{-2}+2a^{-1}b^{-1}-a^{-1}b^{-1}$$

$$=a^{-2}+a^{-1}b^{-1}+b^{-2}.$$

Ex. 8. Divide $x-3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}-y$ by $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+x^{\frac{2}{3}}$.

$$x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}\Big)\,x-3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}-y\,\Big(x^{\frac{1}{3}}-y^{\frac{1}{3}}$$

$$\begin{array}{r} \underline{x-2x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}} \\ -x^{\frac{2}{3}}y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}-y \\ \underline{-x^{\frac{2}{3}}y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}-y} \end{array}$$

$\therefore$ The quotient $=x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

✓Ex. 9. Find the product of

$$\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\left(x^{2^{n-1}}-y^{2^{n-1}}\right).$$

The reqd. product $=\left(x^{2^{n-1}}\right)^2-\left(y^{2^{n-1}}\right)^2=x^{2^{n-1}\times 2^1}-y^{2^{n-1}\times 2^1}$

$$=x^{2^{n-1+1}}-y^{2^{n-1+1}}=x^{2^n}-y^{2^n}.$$

[*N. B.* $2^{n-1}\times 2=2^{n-1+1}=2^n$. So 2^n is the square of 2^{n-1}.

Note the distinction between $x^{2^{n-1}}$ and $(x^2)^{n-1}$.

In $x^{2^{n-1}}$ the power-index of x is 2^{n-1}, but $(x^2)^{n-1}=x^{2n-2}$, the index being $2n-2$.]

✓Ex. 10. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$. [*C. U.*]

$$\frac{x^{2^n}-y^{2^n}}{x^{2^{n-1}}+y^{2^{n-1}}}=\frac{\left(x^{2^{n-1}}\right)^2-\left(y^{2^{n-1}}\right)^2}{x^{2^{n-1}}+y^{2^{n-1}}}$$

$$=\frac{\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\left(x^{2^{n-1}}-y^{2^{n-1}}\right)}{x^{2^{n-1}}+y^{2^{n-1}}}=x^{2^{n-1}}-y^{2^{n-1}}.$$

Ex. 11. Factorize $a+b$.

$$a+b=\left(a^{\frac{1}{3}}\right)^3+\left(b^{\frac{1}{3}}\right)^3=\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right).$$

Ex. 12. Simpilfy $\left(\frac{a^l}{b^m}\right)^n\left(\frac{b^n}{c^l}\right)^m\left(\frac{c^m}{a^n}\right)^l$. [*C. U.* '31]

The given exp. $=\frac{a^{ln}}{b^{mn}}\times\frac{b^{mn}}{c^{lm}}\times\frac{c^{ml}}{a^{nl}}=1$.

Ex. 13. Simplify $(x^a)^{b-c}(x^b)^{c-a}(x^c)^{a-b}$. [*C. U.* '33]

The given exp. $=x^{ab-ac}.\,x^{bc-ab}.\,x^{ca-bc}$

$$=x^{ab-ac+bc-ab+ca-bc}=x^0=1.$$

Ex. 14. Simplify $\left(\frac{x^l}{x^m}\right)^{l+m}\left(\frac{x^m}{x^n}\right)^{m+n}\left(\frac{x^n}{x^l}\right)^{n+l}$.

[*C. U.* '16, '21, '24 ; *D. B.* '25, '30 ; *G. U.* '48, '50]

The given exp- $=(x^{l-m})^{l+m}.\,(x^{m-n})^{m+n}.\,(x^{n-l})^{n+l}$

$$=x^{l^2-m^2}.\,x^{m^2-n^2}.\,x^{n^2-l^2}=x^{l^2-m^2+m^2-n^2+n^2-l^2}=x^0=1.$$

Ex. 15. Simplify

$\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2}\left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2}\left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$. [*C. U.*]

The given exp. $=\left(a^{a-b}\right)^{a^2+ab+b^2}\times\left(x^{b-c}\right)^{b^2+bc+c^2}\times$

$$\left(x^{c-a}\right)^{c^2+ca+a^2}$$

$$=x^{a^3-b^3}.\,x^{b^3-c^3}.\,x^{c^3-a^3}=x^{a^3-b^3+b^3-c^3+c^3-a^3}$$

$=x^0=1.$

Ex. 16. Simplify $\frac{(bc)^{b-c}(ca)^{c-a}(ab)^{a-b}}{(a^{b-c}b^{c-a}c^{a-b})^{-1}}$. [*D. B.* '45]

The given exp. $=b^{b-c}.c^{b-c}.c^{c-a}.a^{c-a}.a^{a-b}.b^{a-b}$

$$\times a^{b-c}.b^{c-a}.c^{a-b}$$

$=a^{a-b+b-c+c-a}.b^{b-c+a-b+c-a}.c^{b-c+c-a+a-b}$

$=a^0.b^0.c^0=1\times1\times1=1.$

[N. B. We know $\frac{1}{a^{-2}}=a^2$. Here the index of the denominator being negative it can be converted into numerator by changing the sign of the index.]

Ex. 17. Simplify $\sqrt[bc]{\frac{x^b}{x^c}}\times\sqrt[ca]{\frac{x^c}{x^a}}\times\sqrt[ab]{\frac{x^a}{x^b}}$.

[*C. U.* '41 ; *W. B. S. F.* '52 ; *D. B.* '49 ; *H. S.* '62 (compl)]

The given exp. $=\sqrt[bc]{x^{b-c}}.\ \sqrt[ca]{x^{c-a}}.\ \sqrt[ab]{x^{a-b}}.$

$$=x^{\frac{b-c}{bc}}.x^{\frac{c-a}{ca}}.x^{\frac{a-b}{ab}}=x^{\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}}$$

$$=x^{\frac{ab-ac+bc-ab+ac-bc}{abc}}=x^{\frac{0}{abc}}=x^0=1.$$

[Here $\sqrt[bc]{x^{b-c}}$ means the bc^{th} root of x^{b-c} and so it is written as $x^{\frac{b-c}{bc}}$.]

Ex. 18. Simplify $\frac{a^{\frac{3}{2}}+ab}{ab-b^3}-\frac{\sqrt{a}}{\sqrt{a}-b}$. [*C. U.* '34]

The given exp. $=\frac{a(a^{\frac{1}{2}}+b)}{b(a-b^2)}-\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}-b}=\frac{a(a^{\frac{1}{2}}+b)}{b(a^{\frac{1}{2}}+b)(a^{\frac{1}{2}}-b)}-\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}-b}$

$$=\frac{a}{b(a^{\frac{1}{2}}-b)}-\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}-b}=\frac{a-a^{\frac{1}{2}}b}{b(a^{\frac{1}{2}}-b)}$$

$$=\frac{a^{\frac{1}{2}}(a^{\frac{1}{2}}-b)}{b(a^{\frac{1}{2}}-b)}=\frac{a^{\frac{1}{2}}}{b}=\frac{\sqrt{a}}{b}.$$

[N. B. Here $a-b^2=(a^{\frac{1}{2}})^2-(b)^2=(a^{\frac{1}{2}}+b)(a^{\frac{1}{2}}-b)$]

Ex. 19. Simplify $\left\{\sqrt[3]{\frac{x^2}{y^4}}\times\sqrt{\frac{y^3}{x^5}}\right\}^{12}\times x^{22}$. [*M. U.* 1894]

The given exp. $=\left\{\frac{x^{\frac{2}{3}}}{y^{\frac{4}{3}}}\times\frac{y^{\frac{3}{2}}}{x^{\frac{5}{2}}}\right\}^{12}\times x^{22}=\frac{x^{\frac{2}{3}\times12}.y^{\frac{3}{2}\times12}}{y^{\frac{4}{3}\times12}.x^{\frac{5}{2}\times12}}\times x^{22}$

$$=\frac{x^8y^{18}\times x^{22}}{y^{16}x^{30}}=\frac{x^{30}y^{18}}{y^{16}x^{30}}=y^{18-16}=y^2.$$

Ex. 20. Simplify $\sqrt[5]{a^8\sqrt{a^8\sqrt{a^{-8}}}}$.

The given exp. $=\sqrt[5]{a^8\sqrt{a^8\times a^{-4}}}=\sqrt[5]{a^8\sqrt{a^4}}=\sqrt[5]{a^8.a^2}$

$$=\sqrt[5]{a^{10}}=a^{\frac{10}{5}}=a^2.$$

Ex. 21. Simplify $\left\{\dfrac{4^{m+\frac{1}{4}}\times\sqrt{2.2^m}}{2\sqrt{2^{-m}}}\right\}^{\frac{1}{m}}$.

[*H. S.* '64 (*C*) ; *C. U.* '47]

The given exp. $=\left\{\dfrac{(2^2)^{m+\frac{1}{4}}\times\sqrt{2^{m+1}}}{2.2^{-\frac{m}{2}}}\right\}^{\frac{1}{m}}=\left\{\dfrac{2^{2m+\frac{1}{2}}.2^{\frac{m+1}{2}}}{2^{1-\frac{m}{2}}}\right\}^{\frac{1}{m}}$

$$=\left\{\frac{2^{2m+\frac{1}{2}+\frac{m+1}{2}}}{2^{\frac{2-m}{2}}}\right\}^{\frac{1}{m}}=\left\{2^{2m+\frac{1}{2}+\frac{m+1}{2}-\frac{2-m}{2}}\right\}^{\frac{1}{m}}$$

$$=\left\{2^{\frac{4m+1+m+1-2+m}{2}}\right\}^{\frac{1}{m}}=\left\{2^{3m}\right\}^{\frac{1}{m}}=2^3=8.$$

Ex. 22. Simplify $\dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.6^n}{15^m.10^{n+2}.6^m}$. [*P. U.* '18]

The given exp $=\dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.2^n.3^n}{3^m.5^m.2^{n+2}.5^{n+2}.2^m.3^m}=\dfrac{2^{m+n+1}.3^{2m}.5^{m+n}}{3^{2m}.2^{m+n+2}.5^{m+n+2}}$

$$=2^{m+n+1-m-n-2}.5^{m+n-m-n-2}=2^{-1}.5^{-2}=\frac{1}{2}\times\frac{1}{5^2}=\frac{1}{50}.$$

[*N. B.* 6^m can be written as $(2\times3)^m$ or $2^m.3^m$]

Ex. 23. Simplify $\dfrac{3^p4^q6^r}{2^{r+q}12^{r+p}}$. [*C. U.* 1935]

The given exp. $=\dfrac{3^p.(2^2)^q.2^r.3^r}{2^{r+q}.(2^2.3)^{r+p}}=\dfrac{3^{p+r}2^{2q+r}}{2^{r+q}.2^{2r+2p}.3^{r+p}}$

$$=\frac{3^{p+r}2^{2q+r}}{2^{3r+2p+q}.3^{r+p}}=2^{2q+r-3r-2p-q}=2^{q-2r-2p}.$$

Ex. 24. Simplify $\dfrac{2^a(2^{a-1})^a}{2^{a+1}\,2^{a-1}}\left\{\dfrac{8^{\frac{a}{3}}}{4}\right\}^{-a}$. [*E. B. S. B.* 1951]

$$\text{The given exp.}=\frac{2^a.2^{a^2-a}}{2^{a+1+a-1}}\cdot\left\{\frac{(2^3)^{\frac{a}{3}}}{2^2}\right\}^{-a}=\frac{2^{a+a^2-a}}{2^{2a}}\cdot\left\{\frac{2^a}{2^2}\right\}^{-a}$$

$$=\frac{2^{a^2}}{2^{2a}}(2^{a-2})^{-a}=2^{a^2-2a}2^{-a^2+2a}=2^{a^2-2a-a^2+2a}=2^0=1.$$

Ex. 25. Simplify $\sqrt[bc]{\dfrac{x^{\frac{b}{c}}}{x^{\frac{c}{b}}}}\;\sqrt[ca]{\dfrac{x^{\frac{c}{a}}}{x^{\frac{a}{c}}}}\;\sqrt[ab]{\dfrac{x^{\frac{a}{b}}}{x^{\frac{b}{a}}}}$. [*C. U.* '38]

$$\text{The given exp.}=\sqrt[bc]{x^{\frac{b}{c}-\frac{c}{b}}}\;\sqrt[ca]{x^{\frac{c}{a}-\frac{a}{c}}}\;\sqrt[ab]{x^{\frac{a}{b}-\frac{b}{a}}}$$

$$=\sqrt[bc]{x^{\frac{b^2-c^2}{bc}}}\;\sqrt[ca]{x^{\frac{c^2-a^2}{ca}}}\;\sqrt[ab]{x^{\frac{a^2-b^2}{ab}}}$$

$$=x^{\frac{b^2-c^2}{b^2c^2}}\cdot x^{\frac{c^2-a^2}{c^2a^2}}\cdot x^{\frac{a^2-b^2}{a^2b^2}}=x^{\frac{b^2-c^2}{b^2c^2}+\frac{c^2-a^2}{c^2a^2}+\frac{a^2-b^2}{a^2b^2}}$$

$$=(x)^{\frac{a^2b^2-a^2c^2+b^2c^2-a^2b^2+a^2c^2-b^2c^2}{a^2b^2c^2}}=x^{\frac{0}{a^2b^2c^2}}=x^0=1.$$

$\left[\text{Note that } \sqrt[bc]{\dfrac{b^2-c^2}{x^{bc}}}=\left(\dfrac{b^2-c^2}{x^{bc}}\right)^{\frac{1}{bc}}=\dfrac{b^2-c^2}{x^{b^2c^2}}\right]$

Ex. ✓26. Simplify $\dfrac{a^2}{(1-a)^n}+\dfrac{2}{(1-a)^{n-1}}-\dfrac{1}{(1-a)^{n-2}}$.

[Here, of the different powers of $(1-a)$, the greatest power is $(1-a)^n$. ∴ the L. C. M. of the denominators is $(1-a)^n$.]

$$\text{The given exp.}=\frac{a^2+2(1-a)-(1-a)^2}{(1-a)^n}=\frac{a^2+2-2a-1+2a-a^2}{(1-a)^n}$$

$$=\frac{1}{(1-a)^n}.$$

[*N. B.* As in division the indices of the powers are subtracted, $\therefore$ on dividing $(1-a)^n$ by $(1-a)^{n-1}$, we have $(1-a)^{n-(n-1)}$ or $(1-a)^1$ or $(1-a)$.]

✓**Ex. 27.** Simplify $\dfrac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}$. [*B.U.* 1891]

$$\text{The given fraction}=\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q+\frac{1}{p}\right)^q\left(q-\frac{1}{p}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}$$

$$=\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^{p+q-p}}{\left(q-\frac{1}{p}\right)^q\left(q+\frac{1}{p}\right)^{q+p-q}}=\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^q}{\left(q-\frac{1}{p}\right)^q\left(q+\frac{1}{p}\right)^p}$$

$$=\frac{\left(\frac{pq+1}{q}\right)^p\left(\frac{pq-1}{q}\right)^q}{\left(\frac{pq-1}{p}\right)^q\left(\frac{pq+1}{p}\right)^p}=\frac{\frac{(pq+1)^p}{q^p}\times\frac{(pq-1)^q}{q^q}}{\frac{(pq-1)^q}{p^q}\times\frac{(pq+1)^p}{p^p}}$$

$$=\frac{(pq+1)^p(pq-1)^q}{q^{p+q}}\times\frac{p^{p+q}}{(pq-1)^q(pq+1)^p}$$

$$=\frac{p^{p+q}}{q^{p+q}}=\left(\frac{p}{q}\right)^{p+q}.$$

✓**Ex. 28.** Simplify :—

$$\frac{1}{1+x^{b-a}+x^{c-a}}+\frac{1}{1+x^{a-b}+x^{c-b}}+\frac{1}{1+x^{a-c}+x^{b-c}}.$$

[*C. U.* '26. '40 ; *D. B.* '29 ; *E. B. S. B.* '52]

Multiplying the numerator and the denominator of the first term by x^a, of the second term by x^b and of the third term by x^c, we have the given expression

$$=\frac{x^a}{x^a+x^b+x^c}+\frac{x^b}{x^b+x^a+x^c}+\frac{x^c}{x^c+x^a+x^b}=\frac{x^a+x^b+x^c}{x^a+x^b+x^c}=1.$$

[*N. B.* In such cases we have to determine by what the numerator and the denominator of each term are to be multiplied, so that all terms may have the same denominator.]

Ex. 29. Simplify :—

$$\frac{1}{1+x^{q-r}+x^{q-p}}+\frac{1}{1+x^{r-p}+x^{r-q}}+\frac{1}{1+x^{p-q}+x^{p-r}}. \quad [P.\ U.\ 1903]$$

The given exp =

$$\frac{x^{-q}}{x^{-q}(1+x^{q-r}+x^{q-p})}+\frac{x^{-r}}{x^{-r}(1+x^{r-p}+x^{r-q})}+\frac{x^{-p}}{x^{-p}(1+x^{p-q}+x^{p-r})}$$

$$=\frac{x^{-q}}{x^{-q}+x^{-r}+x^{-p}}+\frac{x^{-r}}{x^{-r}+x^{-p}+x^{-q}}+\frac{x^{-p}}{x^{-p}+x^{-q}+x^{-r}}$$

$$=\frac{x^{-p}+x^{-q}+x^{-r}}{x^{-p}+x^{-q}+x^{-r}}=1.$$

23. IDENTITIES RELATING TO INDICES

Examples 8

Ex. 1. If $x^y=y^x$, show that $\left(\frac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$; and if also $x=2y$, show that $y=2$. [*C. U.* '28, '49 ; *A. U.* '16 ; *D. B.* '50]

$$\because\ x^y=y^x,\quad \therefore\ x^{\frac{y}{y}}=y^{\frac{x}{y}},\quad \text{or,}\quad x=y^{\frac{x}{y}}.$$

$$\text{Now,}\ \left(\frac{x}{y}\right)^{\frac{x}{y}}=\frac{x^{\frac{x}{y}}}{y^{\frac{x}{y}}}=\frac{x^{\frac{x}{y}}}{x}=x^{\frac{x}{y}-1}.$$

Again, $\because\ x=2y$, and $x^y=y^x$ (hyp.),

$\therefore\ x^y=y^x$, or, $(2y)^y=y^{2y}$, or $(2y)^y=(y^2)^y$,

$\therefore\ y^2=2y$, $\therefore\ y=2$.

[If two equal quantities have the same powers, (other than 0) then their bases must be equal, and if their bases (other than 1) be the same, then their powers must be equal. Thus if $a^x = b^x$, then $a = b$; and if $a^x = a^y$, then $x = y$. In the above sum $(2y)^y$ and $(y^2)^y$ are equal and as they have the same power y, their bases $2y$ and y^2 must be equal.]

Ex. 2. If $m = a^x$, $n = a^y$ and $a^2 = (m^y n^x)^z$,
show that $xyz = 1$. [*H. S.* '63 ; *B. U.* 1890]

Here, $a^2 = (m^y n^x)^z = m^{yz} n^{xz} = (a^x)^{yz} . (a^y)^{xz}$

[putting the values of m and n]

$$= a^{xyz} . a^{xyz} = a^{2xyz}, \quad \therefore \quad a^2 = a^{2xyz},$$

$$\therefore \quad 2xyz = 2, \quad \therefore \quad xyz = 1.$$

Ex. 3. If $a^x = b$, $b^y = c$, $c^z = a$, prove that $xyz = 1$.
[*C. U.* 1956]

Here, $a = c^z = (b^y)^z = b^{yz} = (a^x)^{yz} = a^{xyz}$,

or, $a^1 = a^{xyz}$, $\therefore \quad xyz = 1$.

Ex. 4. If $a^x = b^y$ and $b^x = a^y$, then $a = b$.

$\because \quad a^x = b^y$......(1) and $a^y = b^x$......(2)

$\therefore$ From (1)×(2) we have $a^{x+y} = b^{x+y}$, $\therefore \quad a = b$.

Ex. 5. If $x^{\frac{1}{a}} = y^{\frac{1}{b}} = z^{\frac{1}{c}}$ and $xyz = 1$, prove that $a + b + c = 0$.
[*H. S.* '64]

$$x^{\frac{1}{a}} = y^{\frac{1}{b}} = z^{\frac{1}{c}} = k \text{ (Suppose)}$$

$\therefore \quad \left(x^{\frac{1}{a}}\right)^a = k^a$, or, $x = k^a$. Similarly $y = k^b$ and $z = k^c$.

Again, $\because \quad xyz = 1$, $\therefore \; k^a . k^b . k^c = 1$, or, $k^{a+b+c} = 1 = k^0$.

$$\therefore \quad a + b + c = 0.$$

Ex. 6. If $a=x^{q+r}y^{p}$, $b=x^{r+p}y^{q}$, $c=x^{p+q}y^{r}$,
show that $a^{q-r}.\,b^{r-p}.\,c^{p-q}=1$. [*C. U.* '51]

$a^{q-r}.b^{r-p}.c^{p-q}=(x^{q+r}y^{p})^{q-r}.(x^{r+p}y^{q})^{r-p}.(x^{p+q}y^{r})^{p-q}$

$=x^{q^2-r^2}.y^{pq-pr}.x^{r^2-p^2}.y^{qr-pq}.x^{p^2-q^2}.y^{rp-rq}$

$=x^{q^2-r^2+r^2-p^2+p^2-q^2}.y^{pq-pr+qr-pq+pr-qr}$

$=x^{0}.y^{0}=1\times1=1.$

Ex. 7. If $x=3^{\frac{1}{3}}-3^{-\frac{1}{3}}$, prove that $3x^3+9x=8$. [*Pat.* '30]

Here $x^3=\left(3^{\frac{1}{3}}-3^{-\frac{1}{3}}\right)^3=\left(3^{\frac{1}{3}}\right)^3-\left(3^{-\frac{1}{3}}\right)^3-3.3^{\frac{1}{3}}3^{-\frac{1}{3}}\left(3^{\frac{1}{3}}-3^{-\frac{1}{3}}\right)$

or, $x^3=3-3^{-1}-3.3^{0}(x)$ $\left[\because\ 3^{\frac{1}{3}}-3^{-\frac{1}{3}}=x\right]$

or, $x^3=3-\frac{1}{3}-3x$, or, $x^3=\frac{8}{3}-3x$, or, $3x^3=8-9x$,

$\therefore\ 3x^3+9x=8.$

Ex. 8. If $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, prove that $x^3-6x^2+6x-2=0$.
[*C. U.* '30, '36, '42, '50 ; *D. B.* '42]

$\because\ x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, $\therefore\ x-2=2^{\frac{2}{3}}+2^{\frac{1}{3}}$,

$\therefore\ (x-2)^3=\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}\right)^3$

or, $x^3-6x^2+12x-8=\left(2^{\frac{2}{3}}\right)^3+\left(2^{\frac{1}{3}}\right)^3+3.2^{\frac{2}{3}}.2^{\frac{1}{3}}\left(2^{\frac{2}{3}}+2^{\frac{1}{3}}\right)$

$=2^2+2+3.2.(x-2)$ $\left[\because\ x-2=2^{\frac{2}{3}}+2^{\frac{1}{3}}\right]$

or, $x^3-6x^2+12x-8=6+6x-12$

or, $x^3-6x^2+12x-8-6-6x+12=0$

$\therefore\ x^3-6x^2+6x-2=0.$

Ex. 9. If $a^{m^n}=(a^m)^n$, find m in terms of n. [*Pat.* '18]

$\because\ a^{m^n}=(a^m)^n$, $\therefore\ a^{m^n}=a^{mn}$, $\therefore\ m^n=mn$,

or, $\frac{m^n}{m}=n$, or, $m^{n-1}=n$, $\therefore\ m=n^{\frac{1}{n-1}}.$

Ex. 10. If $p^x=q^y=r^z$ and $q^2=pr$, show that $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.

[First express p and r in terms of q]

$$\because\ p^x=q^y,\quad \therefore\ p=q^{\frac{y}{x}}, \text{ and } \because r^z=q^y, \therefore r=q^{\frac{y}{z}}$$

$$\text{Again, } \because\ q^2=pr,\quad \therefore\ q^2=q^{\frac{y}{x}}.q^{\frac{y}{z}}=q^{\frac{y}{x}+\frac{y}{z}}$$

$$\therefore\ \frac{y}{x}+\frac{y}{z}=2,\qquad \therefore\ \frac{1}{x}+\frac{1}{z}=\frac{2}{y}\quad [\text{ dividing by } y\ ]$$

Ex. 11. Prove that $(x+y)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)$

$$=\frac{x^{2^n}-y^{2^n}}{x-y}.\qquad [C.\ U.\ 1945]$$

$$\text{The left side}=\frac{(x-y)(x+y)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

$$=\frac{(x^2-y^2)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

$$=\frac{(x^4-y^4)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

Thus multiplying up to the last we have

$$\text{the left hand side}=\frac{\left(x^{2^{n-1}}-y^{2^{n-1}}\right)\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

$$=\frac{\left(x^{2^{n-1}}\right)^2-\left(y^{2^{n-1}}\right)^2}{x-y}=\frac{x^{2^n}-y^{2^n}}{x-y}.$$

[For **Exponential Equations** see the chapter on Quadratic Equations.]

Exercise 6

Find the value of the following :—

1. $27^{\frac{2}{3}}$ 2. $9^{-\frac{3}{2}}$ 3. $(\frac{1}{81})^{-\frac{5}{4}}$ 4. $\sqrt[5]{32^4}$ [*P. U.* 1903]

5. $243^{-\frac{2}{5}}$ 6. $\sqrt[4]{625^3}$.

Simplify the following :—

7. $x^{-\frac{2}{3}} \div x^{\frac{1}{3}} y^{-\frac{1}{4}}$ 8. $\left\{\left(x^{-3}\right)^{\frac{2}{3}}\right\}^{-\frac{1}{2}}$.

9. $8a^{-\frac{3}{4}} \div 6b^{-\frac{2}{3}} \times \frac{1}{2}ab$ 10. $x^{\frac{1}{3}}.x^{-\frac{1}{4}}.x^{\frac{1}{2}} \div x^{\frac{7}{12}}$,

10. $\left(a^3b^2\right)^{-\frac{1}{6}} \times \left(a^2\right)^{\frac{1}{3}} \div \left(b^3\right)^{\frac{1}{9}}$. 12. $\dfrac{\frac{1}{4^{-3}} - \frac{2}{10^{-2}}}{\frac{1}{2^{-2}} + \frac{1}{4^{-1}}}$

13. $\sqrt[4]{a^2b^2c^4} \div \sqrt[3]{a^4b^6} \times \left(a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{1}{4}}\right)^{-4}$.

14. $\sqrt{a^{-5}b^6} \times \sqrt[4]{a^2.b^{-3}}$. 15. $\sqrt[3x]{a^4} \div \sqrt[5x]{a^{-4}}$

16. $\sqrt[4]{a^2.\sqrt[3]{a^{-2}}}$. 17. $\left(\dfrac{x^{-3}y^4}{x^2y^{-1}}\right)^{\frac{3}{5}} \div \left(\dfrac{x^2y^3}{x^{-3}y^4}\right)^2$.

17. (*a*). $\left\{\sqrt[3]{4} \times \dfrac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right\}^{\frac{1}{4}}$ [*Pat.* '23]

18. (i) Multiply $x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 1$ by $x^{\frac{1}{4}} + x^{-\frac{1}{4}} - 1$; [*C. U.* '33]
and $x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}} + y^{-\frac{1}{2}}$. [*Pat.* '29]

(ii) Divide $a^2 + 6ac^{\frac{1}{3}} - 4b + 9c^{\frac{2}{3}}$ by $a - 2b^{\frac{1}{2}} + 3c^{\frac{1}{3}}$.

19. Find the product of $(x^n - y^{-n})(x^n + y^{-n})$.

20. Multiply $a^{-m} + b^n$ by $a^{-2m} - a^{-m}b^n + b^{2n}$.

21. Divide $x^{3^n} - y^{3^n}$ by $x^{3^{n-1}} - y^{3^{n-1}}$.

22. Resolve into factors $a-b$, if it is the difference of two squares or cubes.

23. Factorize $x^{2m} - y^{2n}$.

Simplify the following :—

24. $\dfrac{x^{a+b}.x^{a-b}.x^{c-2a}}{x^{c-a}}$. [*C. U.* 1870]

25. $(a+b)^m \times (a-b)^m \times (a^2+b^2)^m$. [*M. U.* 1889]

26. $\left(a^{n^2}-1\right)^{\frac{n}{n+1}} \times \dfrac{\sqrt{a^{2n}}}{a}$.

27. $\{(x^{a+b-c} \times x^{a-b+c})^b\}^c$. [*M. U.*]

28. $\left(\dfrac{x^a}{x^b}\right)^{a+b} \div \left(\dfrac{x^a}{x^{a-b}}\right)^{\frac{a^2}{b}}$ [*M. U.* 1890]

29. $\left(\dfrac{x^p}{x^q}\right)^{p+q} \div \left(\dfrac{x^{p+q}}{x^{p-q}}\right)^{\frac{p^2}{q}}$ [*C. U* 1902]

30. $\left(\dfrac{x^m}{x^n}\right)^{m+n-l} \times \left(\dfrac{x^n}{x^l}\right)^{n+l-m} \times \left(\dfrac{x^l}{x^m}\right)^{l+m-n}$. [*Pat.* '28]

31. $\left\{\dfrac{\left(9^{n+\frac{1}{4}}\right)\sqrt{3.3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}$. [*C. U.* '25 ; *W. B S. F.* '58]

32. $\left(\dfrac{x^{m^2+n^2}}{x^{-mn}}\right)^{m-n} \times \left(\dfrac{x^{n^2+l^2}}{x^{-nl}}\right)^{n-l} \times \left(\dfrac{x^{l^2+m^2}}{x^{-lm}}\right)^{l-m}$

32. (a) $\sqrt[ln]{\dfrac{x^l}{x^n}} \times \sqrt[mn]{\dfrac{x^n}{x^m}} \times \sqrt[lm]{\dfrac{x^m}{x^l}}$ [*H. S.* '61]

33. $\left(\dfrac{b+c}{x^{c-a}}\right)^{\frac{1}{a-b}} \times \left(\dfrac{c+a}{x^{a-b}}\right)^{\frac{1}{b-c}} \times \left(\dfrac{a+b}{x^{b-c}}\right)^{\frac{1}{c-a}}$.

✓34. $\dfrac{(p+\frac{1}{q})^m(p-\frac{1}{q})^n}{(q+\frac{1}{p})^m(q-\frac{1}{p})^n}$. [*B. U.*]

35. $\sqrt[a-c]{x^{\frac{1}{a-b}}} \times \sqrt[b-a]{x^{\frac{1}{b-c}}} \times \sqrt[c-b]{x^{\frac{1}{c-a}}}$.

36. $\dfrac{2^{m+2}.3^{2m-n}.5^{m+n+2}.6^n}{6^m.10^{n+2}.15^m}$. [*G. U.* '51]

37. $\dfrac{2^n.6^{m+1}.10^{m-n}.15^{m+n-2}}{4^m.3^{2m+n}.25^{m-1}}$. ✓38. $\dfrac{e^{2x}+e^{-x}-e^x-1}{e^{2x}-e^{-x}+e^x-1}$.

✓39. $\dfrac{1}{1+x^{m-n}+x^{m-p}}+\dfrac{1}{1+x^{n-p}+x^{n-m}}+\dfrac{1}{1+x^{p-m}+x^{p-n}}$.

[*C. U.* '37, '42 ; *G. U.* 49 ; *E. B. S. B* '51]

40. $(1+x^{a-b}+x^{a-c})^{-1}+(1+x^{b-c}+x^{b-a})^{-1}$
$+(1+x^{c-a}+x^{c-b})^{-1}$.

41. $\dfrac{\left\{(a^m)^{\frac{1}{r}}(a^q)^{\frac{1}{n}}\right\}^{nr}}{\left\{\sqrt[q]{b^n}(\sqrt[m]{b})^r\right\}^{mq}}\div\left\{\left(\dfrac{a}{b}\right)^q\right\}^r$. [*D. B.* '23, '50]

42. $\dfrac{1}{(1-x)^{\frac{3}{2}}}-\dfrac{x+1}{(1-x)^{\frac{1}{3}}}$.

42.. (*a*) $\left(a^{\frac{1}{x-y}}\right)^{\frac{1}{x-z}}\times\left(a^{\frac{-1}{y-z}}\right)^{\frac{1}{y-x}}\times\left(a^{\frac{1}{z-x}}\right)^{\frac{1}{z-y}}$. [*W.B.S.F.* '53]

43. $\dfrac{1}{(4x^3-3x)^2}-\left\{\dfrac{\dfrac{3\sqrt{1-x^2}}{x}-\dfrac{(1-x^2)^{\frac{3}{2}}}{x^3}}{1-3\left(\dfrac{1-x^2}{x^2}\right)}\right\}^2$ [*C.U.*]

44.✓ If $m^n=n^m$, show that $\left(\dfrac{m}{n}\right)^{\frac{m}{n}}=m^{\frac{m}{n}-1}$.

45. If $x^2=y^3$, prove that $\left(\dfrac{x}{y}\right)^{\frac{3}{2}}+\left(\dfrac{y}{x}\right)^{\frac{2}{3}}=x^{\frac{1}{2}}+y^{-\frac{1}{3}}$.

46. If $a^x=z^y$ and $a^z=z^x$, then $x^2=yz$.

47.✓ If $\left(x^{n^2}\right)^n=\left(x^{2^n}\right)^2$, then $\sqrt[n+1]{n^3}=2$.

48. If $p=a^x$, $q=a^y$ and $(p^yq^x)^z=a^2$, show that $xyz=1$.
[*C. U*, '29, 50 ; *D. B.* '37 ; *Pat.* '19, '21]

49. If $a^{\frac{1}{x}}=b^{\frac{1}{y}}=c^{\frac{1}{z}}$ and $abc=1$, prove that $x+y+z=0$.

50. If $x=2^{\frac{1}{3}}+2^{-\frac{1}{3}}$, show that $2x^3-6x=5$.

51. If $x=1+3^{\frac{2}{3}}+3^{\frac{1}{3}}$, prove that $x^3-3x^2-6x-4=0$.

52. If $y=x^{\frac{1}{3}}-x^{-\frac{1}{3}}$, then $y^3+3y=x-\frac{1}{x}$.

53. If $x=2^{\frac{2}{3}}+2^{\frac{1}{3}}$, show that $x^3-6x=6$. [*A. U.* '12]

54. If $x^a=y^b=z^c$ and $y^2=xz$, prove that $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$.

55. If $a=xy^{p-1}$, $b=xy^{q-1}$, $c=xy^{r-1}$, show that a^{q-r}, b^{r-p}, $c^{p-q}=1$. [*Pat. U.* 1918]

✓56. If $x^{p^q}=\left(x\sqrt{p}\right)^q$, find p in terms of q.

SURDS

24. Incommensurable quantity : We know that it is not always possible to express two quantities of the same kind by a common unit.

As for example, if the side of a square is 1 inch, its diagonal is $\sqrt{2}$ inches. Here we find that the numerical value of $\sqrt{2}$ can never be *exactly* expressed as a multiple or fraction of unity (*i.e.*, 1), though its value to any desired number of decimal places can be found. Such a quantity is said to be **incommensurable.**

25. Surds : When a root of a quantity (a number or expression) cannot be *exactly* determined, the root is called a **surd.**

Thus (*i*), $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt{a^2+b^2}$ are surds.

But $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{\frac{9}{16}}$, $\sqrt{x^2+2xy+y^2}$, though surds in form, are not really surds. For they may be exactly expressed without the radical (root) sign.

Thus $\sqrt{4}=2$, $\sqrt[3]{27}=3$, $\sqrt{\frac{9}{16}}=\frac{3}{4}$, $\sqrt{x^2+2xy+y^2}=x+y$.

(*ii*) Algebraical quantities, such as $\sqrt{a}$, $\sqrt[3]{x}$ are also called surds, although the values of a and x may be such that they are not really surds.

26. Rational and Irrational quantities : Quantities that can be expressed as the ratio of two integers are called **rational** quantities ; and quantities that cannot be so expressed are called **irrational** quantities.

(*i*) Thus 3, $\frac{2}{5}$, $\sqrt{4}$, $\sqrt{a^2+b^2+2ab}$ are rational quantities.

(*ii*) $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{x^3}$ are irrational quantities.

[*N. B.* When a radical sign (*i.e.*, a root symbol) is put before an arithmetical number, it denotes only the arithmetical root ; but the root symbol, put before an algebraical quantity, denotes one of the roots only. As for instance. $\sqrt{x}$ has two values (one positive and one negative), but $\sqrt{7}$ denotes the positive arithmetical root only, unless it is expresssly written as $\pm\sqrt{7}$.]

27. Pure and Mixed Surds : Surds which have no rational factors are called **pure** surds. Thus $\sqrt{5}$, $\sqrt[3]{x}$ are pure surds.

Surds which consist of a rational factor and a surd factor are called **mixed** surds. Thus, $3\sqrt{2}$, $a\sqrt{b}$ are mixed surds.

28. Simple and Compound Surds : When a surd consists of one term only, it is called a **simple** surd.

The algebraic sum of two or more surds is called a **compound** surd.

Thus $\sqrt{3}$, $2\sqrt{5}$ are simple surds, while $\sqrt{3}+2\sqrt{5}$, $\sqrt{7}-3\sqrt{2}$ are compound surds.

The algebraic sum of two surds or a rational quantity and a surd is said to be a **binomial** surd. Thus $2+\sqrt{3}$, $2\sqrt{3}+\sqrt{5}$, $3\sqrt{7}-\sqrt[3]{4}$ are bionomial surds.

Similarly $\sqrt{5}+\sqrt{3}+\sqrt{2}$, $3+\sqrt{7}-\sqrt{2}$ are **trinomial** surds.

29. Order of Surds : The **order** of a surd is indicated by its root symbol. So the order is determined by the fractional index of the root.

Thus (*i*) $\sqrt{3}$ and $2^{\frac{1}{2}}$ are surds of the **second** order or **quadratic** surds.

(*ii*) $\sqrt[3]{4}$, $5^{\frac{2}{3}}$, $x^{\frac{1}{3}}$ are surds of the **third** order or **cubic** surds.

Similarly, $\sqrt[4]{7}$, $\sqrt[n]{x}$ are surds of the 4th and *n*th orders respectively.

30. **Equiradical Surds :** Surds of the same order are called equiradical surds.

Surds are said to be of the same order, when their surd indices are the same, *i.e.*, when the same root is required to be taken.

Surds are of **different** orders when their surd indices are different. Thus, (*i*) $\sqrt{x}$, $x^{\frac{2}{4}}$, $x^{\frac{3}{6}}$ are surds of the same order, (*ii*) while $\sqrt{a}$ and $\sqrt[3]{a}$ are surds of different orders.

$\sqrt{\ }$, $\sqrt[3]{\ }$, etc, are said to be radical signs.

31. **Similar and Dissimilar Surds :**

Two or more surds are said to be **similar** or **like** when they have or can be reduced to have the same irrational or surd factor.

Surds which are not similar or like are said to be **dissimilar** or **unlike** surds.

Thus (*i*) $\sqrt{3}$, $2\sqrt{3}$, $5\sqrt{3}$ are similar surds.

$\sqrt{8}$ and $\sqrt{18}$ are also two similar surds, because they can be reduced to $2\sqrt{2}$ and $3\sqrt{2}$ respectively.

(*ii*) $\sqrt{3}$ and $\sqrt{5}$, $\sqrt{5}$ and $\sqrt[3]{5}$, $\sqrt{20}$ and $\sqrt{75}$ are dissimilar surds.

Rational quantity expressed as a Surd :

Any rational quantity can be expressed in the form of a surd of any order.

Thus, $3=\sqrt{9}=\sqrt[3]{27}=\sqrt[4]{81}=\sqrt[n]{3^n}\cdots,$

$$x=\sqrt{x^2}=\sqrt[3]{x^3}=\sqrt[4]{x^4}=\sqrt[n]{x^n}\cdots$$

Reduction of surds : Surds of different orders can be reduced to surds of the same order.

Example : Reduce $\sqrt[3]{x^2}$ and $\sqrt[4]{x^3}$ to surds of the same order.

Here the given surds are of the 3rd. and 4th. orders respectively. The L. C. M. of 3 and 4 is 12. So the common order of the two will be equal to 12.

Now, $\sqrt[3]{x^2}=x^{\frac{2}{3}}=x^{\frac{2\times4}{3\times4}}=x^{\frac{8}{12}}=\sqrt[12]{x^8}$,

and $\sqrt[4]{x^3}=x^{\frac{3}{4}}=x^{\frac{3\times3}{4\times3}}=x^{\frac{9}{12}}=\sqrt[12]{x^9}$.

Thus the two given surds are reduced to surds of the same order 12.

Comparison of surds : In order to compare the values of two or more surds, they should be reduced to surds of the same order.

Ex. 1. Which is greater $\sqrt{5}$ or $\sqrt[3]{9}$?

Here the L. C. M. of 2 and $3=6$.

Now, $\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}$,

and $\sqrt[3]{9}=9^{\frac{1}{3}}=9^{\frac{2}{6}}=\sqrt[6]{9^2}=\sqrt[6]{81}$.

$\because$ $\sqrt[6]{125}$ is greater than $\sqrt[6]{81}$, $\therefore$ $\sqrt{5}$ is greater than $\sqrt[3]{9}$.

Ex. 2. Arrange $\sqrt[3]{3}$, $\sqrt{2}$ and $\sqrt[4]{8}$ in descending order of magnitude.

Here the indices of the root are 3, 2, 4 respectively and their L. C. M. $=12$.

Now, $\sqrt[3]{3}=\sqrt[12]{3^4}=\sqrt[12]{81}$, $\sqrt{2}=\sqrt[12]{2^6}=\sqrt[12]{64}$,

$\sqrt[4]{8}=\sqrt[12]{8^3}=\sqrt[12]{512}$.

$\therefore$ When arranged in the descending order of magnitude, they are $\sqrt[4]{8}$, $\sqrt[3]{3}$, $\sqrt{2}$.

33. The simplest form of surds :

When a surd is expressed as the product of a rational factor and an irrational factor, it is said to be in its simplest form.

Again, the product of a rational quantity and an irrational quantity can be expressed as a complete surd.

Ex. 1. Reduce $\sqrt{128}$ and $\sqrt[3]{192}$ to the simplest form.

$\sqrt{128}=\sqrt{64\times 2}=\sqrt{8^2\times 2}=(8^2\times 2)^{\frac{1}{2}}=(8^2)^{\frac{1}{2}}\times(2)^{\frac{1}{2}}=8\times 2^{\frac{1}{2}}=8\sqrt{2}$,

$\sqrt[3]{192}=\sqrt[3]{64\times 3}=\sqrt[3]{4^3\times 3}=4\sqrt[3]{3}$.

Ex. 2. Express $3\sqrt{2}$ and $2\sqrt[3]{3}$ as complete surds,

Now, $3\sqrt{2}=\sqrt{3^2}\times\sqrt{2}=\sqrt{9}\times\sqrt{2}=9^{\frac{1}{2}}\times 2^{\frac{1}{2}}=(9\times 2)^{\frac{1}{2}}=18^{\frac{1}{2}}=\sqrt{18}$;

$2\sqrt[3]{3}=\sqrt[3]{2^3}\times\sqrt[3]{3}=\sqrt[3]{8}\times\sqrt[3]{3}=\sqrt[3]{8\times 3}=\sqrt[3]{24}$.

34. Addition and subtraction of surds :

To find the sum of two or more surds, they should first be reduced to their simplest forms. If they are similar, the product of the sum of their rational factors and the irrational factor is their required sum.

The difference of two surds is found in the same way.

Example 1. Find the sum of $4\sqrt{2}$, $3\sqrt{2}$ and $6\sqrt{2}$.

Here the sum of the rational factors $=4+3+6=13$.

The required sum $=(4+3+6)\times\sqrt{2}=13\sqrt{2}$.

Ex. 2. Subtract $\sqrt{32}$ from $7\sqrt{2}$.

$7\sqrt{2}-\sqrt{32}=7\sqrt{2}-\sqrt{16\times 2}=7\sqrt{2}-4\sqrt{2}=(7-4)\sqrt{2}=3\sqrt{2}$.

Ex. 3. Add together $\sqrt{12}$, $-4\sqrt{3}$ and $\sqrt{108}$.

The sum $=\sqrt{12}-4\sqrt{3}+\sqrt{108}=\sqrt{4\times 3}-4\sqrt{3}+\sqrt{36\times 3}$.

$=2\sqrt{3}-4\sqrt{3}+6\sqrt{3}=(2-4+6)\sqrt{3}=4\sqrt{3}$.

Note : If the given surds are dissimilar or unlike, their sum or difference is found by collecting them with the plus (+) or minus (−) sign.

Thus (*i*) the sum of $\sqrt{3}$ and $2\sqrt{5}$ is $\sqrt{3}+2\sqrt{5}$.

(*ii*) the sum of $2\sqrt{3}$, $3\sqrt{2}$ and $4\sqrt{3}$

$$=2\sqrt{3}+3\sqrt{2}+4\sqrt{3}=(2+4)\sqrt{3}+3\sqrt{2}=6\sqrt{3}+3\sqrt{2}.$$

(*iii*) the difference of $3\sqrt{5}$ and $2\sqrt{3}=3\sqrt{5}-2\sqrt{3}$.

35. Multiplication of Simple Surds :

(*i*) If the simple surds are of the same order, their product is obtained by multiplying their rational factors and irrational factors separately.

Thus, (1) $3\sqrt{2}\times 4\sqrt{5}=(3\times 4)\times(\sqrt{2}\times\sqrt{5})=12\sqrt{10}$.

(2) $\sqrt{3}\times\sqrt{3}=\sqrt{9}=3$.

(3) $4\sqrt{3}\times 5\sqrt{6}=(4\times 5)\times(\sqrt{3}\times\sqrt{6})=20\sqrt{18}=20\sqrt{9.2}$

$$=20\times 3\sqrt{2}=60\sqrt{2}.$$

(4) $2\sqrt{5}\times\sqrt{32}\times 3\sqrt{2}=2\sqrt{5}\times\sqrt{16.2}\times 3\sqrt{2}$

$$=2\sqrt{5}\times 4\sqrt{2}\times 3\sqrt{2}=2.4.3\sqrt{5.2.2}$$

$$=24.2\sqrt{5}=48\sqrt{5}.$$

(*ii*) If the simple surds be of different orders, they should first be reduced to the surds of the same order and then their product is obtained as stated above in (*i*).

Thus, $3\sqrt[3]{3}\times 2\sqrt[4]{2}=3\sqrt[12]{3^4}\times 2\sqrt[12]{2^3}$

$$=3\times 2\times\sqrt[12]{3^4\times 2^3}=6\sqrt[12]{648}.$$

(*iii*) Multiplication of compound surds :

Compound surds are multiplied together in the same way as compound algebraical expressions. As for example :

Ex. 1. Multiply $2\sqrt{3}+3\sqrt{4}$ by $3\sqrt{2}+\sqrt{3}$.

The product $=(2\sqrt{3}+3\sqrt{4})(3\sqrt{2}+\sqrt{3})$

$=(2\sqrt{3}\times 3\sqrt{2})+(3\sqrt{4}\times 3\sqrt{2})+(2\sqrt{3}\times\sqrt{3})+(3\sqrt{4}\times\sqrt{3})$

$=6\sqrt{6}+9\sqrt{8}+6+3\sqrt{12}=6\sqrt{6}+9\sqrt{4.2}+6+3\sqrt{4.3}$

$=6\sqrt{6}+18\sqrt{2}+6+6\sqrt{3}$.

Ex. 2. Find the product of $3\sqrt{2}+2\sqrt{5}$ and $2\sqrt{6}-4\sqrt{3}$.

The required product $=(3\sqrt{2}+2\sqrt{5})(2\sqrt{6}-4\sqrt{3})$

$=(3\sqrt{2}\times 2\sqrt{6})+(2\sqrt{5}\times 2\sqrt{6})-(3\sqrt{2}\times 4\sqrt{3})-(2\sqrt{5}\times 4\sqrt{3})$

$=6\sqrt{12}+4\sqrt{30}-12\sqrt{6}-8\sqrt{15}$

$=12\sqrt{3}+4\sqrt{30}-12\sqrt{6}-8\sqrt{15}.$

36. Rationalisation of surds :

(*a*) To rationalise a surd is to express it in a rational form multiplying it by a suitable surd.

This suitable surd factor is called the rationalising factor. Thus, if the product of two surds is rational, each surd is called a **rationalising factor** of the other.

Ex. 1. If $3\sqrt{5}$ is multiplied by $\sqrt{5}$, we get 3.5 or 15, which is rational. Hence $3\sqrt{5}$ is rationalised when it is multiplied by $\sqrt{5}$ and here $\sqrt{5}$ is the rationalising factor.

Ex. 2. $2\sqrt{8}$ is rationalised when it is multiplied by $\sqrt{2}$, for $2\sqrt{8}\times\sqrt{2}=2\sqrt{16}=2.4=8$. Here $\sqrt{2}$ is the rationalising factor.

Ex. 3. $\sqrt[3]{4}=\sqrt[3]{2^2}=2^{\frac{2}{3}}$; and $2^{\frac{2}{3}}\times 2^{\frac{1}{3}}=2$, which is rational. Hence $2^{\frac{1}{3}}$ or $\sqrt[3]{2}$ is the rationalising factor of $\sqrt[3]{4}$.

Ex. 4. $\because\ (\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=(\sqrt{3})^2-(\sqrt{2})^2=3-2=1,$
$\therefore\ \sqrt{3}+\sqrt{2}$ is rationalised when multiplied by $\sqrt{3}-\sqrt{2}$. Hence $\sqrt{3}-\sqrt{2}$ is the rationalising factor of $\sqrt{3}+\sqrt{2}$.

(*b*) If the denominator of a fraction be a surd, it can be reduced to a fraction with rational denominator by multiplying both the denominator and the numerator by the rationalising factor of the denominator.

Ex. 1. $\dfrac{3}{2\sqrt{2}}=\dfrac{3\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\dfrac{3\sqrt{2}}{4}.$

Ex. 2. $\dfrac{2}{\sqrt[3]{3}}=\dfrac{2\sqrt[3]{3^2}}{\sqrt[3]{3}\times\sqrt[3]{3^2}}=\dfrac{2\sqrt[3]{9}}{\sqrt[3]{3^3}}=\dfrac{2\sqrt[3]{9}}{3}.$

Ex. 3. Express $\frac{3+\sqrt{2}}{3-\sqrt{2}}$ with a rational denominator.

$$\frac{3+\sqrt{2}}{3-\sqrt{2}}=\frac{(3+\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{(3+\sqrt{2})^2}{(3)^2-(\sqrt{2})^2}=\frac{9+2+6\sqrt{2}}{9-2}$$

$$=\frac{11+6\sqrt{2}}{7}.$$

Ex. 4. $$\frac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}=\frac{(3\sqrt{3}-2\sqrt{2})(3\sqrt{3}-2\sqrt{2})}{(3\sqrt{3}+2\sqrt{2})(3\sqrt{3}-2\sqrt{2})}$$

$$=\frac{(3\sqrt{3}-2\sqrt{2})^2}{(3\sqrt{3})^2-(2\sqrt{2})^2}=\frac{27+8-12\sqrt{6}}{27-8}=\frac{35-12\sqrt{6}}{19}.$$

Ex. 5. Rationalise the denominator of the fraction

$$\frac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}.$$

The given fraction $$=\frac{(\sqrt{a+1}+\sqrt{a-1})^2}{(\sqrt{a+1}-\sqrt{a-1})(\sqrt{a+1}+\sqrt{a-1})}$$

$$=\frac{a+1+a-1+2\sqrt{a^2-1}}{(a+1)-(a-1)}=\frac{2a+2\sqrt{a^2-1}}{2}=a+\sqrt{a^2-1}.$$

37. Division of surds: To divide one surd by another (simple or compound) the quotient is to be expressed as a fraction with a rational denominator as shown above.

Ex. 1. Divide $\sqrt{3}$ by $\sqrt{5}$.

$$\sqrt{3}\div\sqrt{5}=\frac{\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{3}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{\sqrt{15}}{5}.$$

Ex. 2. Divide $\sqrt{a}+2$ by $\sqrt{a}-1$.

$$\frac{\sqrt{a}+2}{\sqrt{a}-1}=\frac{(\sqrt{a}+2)(\sqrt{a}+1)}{(\sqrt{a}-1)(\sqrt{a}+1)}=\frac{a+3\sqrt{a}+2}{a-1}$$

38. Conjugate or Complementary surds :

Two **binomial quadratic** surds are said to be **conjugate** (or **complementary**) to each other if they differ only in the sign connecting their terms.

Thus, $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$, or $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are conjugate surds.

Hence the product of two conjugate surds are always rational and evidently each is the rationalising factor of the other.

Examples 9

Ex. 1. Find the value of $3\sqrt{48}-4\sqrt{75}+5\sqrt{192}$. [*C. U.* '47]

The given exp. $=3\sqrt{16\times3}-4\sqrt{25\times3}+5\sqrt{64\times3}=3\times4\sqrt{3}$

$$-4\times5\sqrt{3}+5\times8\sqrt{3}=12\sqrt{3}-20\sqrt{3}+40\sqrt{3}=32\sqrt{3}.$$

Ex. 2. Prove that $\sqrt{175}-\sqrt{112}=\sqrt{7}$. [*C. U.* '32]

$$\sqrt{175}-\sqrt{112}=\sqrt{25\times7}-\sqrt{16\times7}=5\sqrt{7}-4\sqrt{7}=\sqrt{7}.$$

Ex. 3. Given $\sqrt{2}=1{\cdot}4142$, find to three decimal places the value of $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$. [*H. S.* 1960 (Compl.)]

$$\frac{1+\sqrt{2}}{3-2\sqrt{2}}=\frac{(1+\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}=\frac{3+5\sqrt{2}+4}{(3)^2-(2\sqrt{2})^2}$$

$$=\frac{7+5\sqrt{2}}{9-8}=7+5\sqrt{2}$$

$$=7+5\times1{\cdot}4142=7+7{\cdot}0710=14{\cdot}071.$$

Ex. 4. Bring $\frac{7}{\sqrt{2}+\sqrt[4]{2}+1}$ to a form with a rational denominator. [*B. U.*]

The given fraction $=\frac{7(\sqrt{2}+1-\sqrt[4]{2})}{(\sqrt{2}+1+\sqrt[4]{2})(\sqrt{2}+1-\sqrt[4]{2})}$

$$=\frac{(7\sqrt{2}+1-\sqrt[4]{2})}{(\sqrt{2}+1)^2-(\sqrt[4]{2})^2}=\frac{7(\sqrt{2}+1-\sqrt[4]{2})}{3+2\sqrt{2}-\sqrt{2}}=\frac{7(\sqrt{2}-\sqrt[4]{2}+1)}{3+\sqrt{2}}$$

$$=\frac{7(3-\sqrt{2})(\sqrt{2}-\sqrt[4]{2}+1)}{3^2-(\sqrt{2})^2}=\frac{7(3-\sqrt{2})(\sqrt{2}-\sqrt[4]{2}+1)}{7}$$

$$=(3-\sqrt{2})(\sqrt{2}-\sqrt[4]{2}+1).$$

Ex. 5. Show that $\dfrac{a\sqrt{a+x}}{\sqrt{a+x}-\sqrt{x}}=a+x+\sqrt{ax+x^2}$. [*B.U.*]

$$\frac{a\sqrt{a+x}}{\sqrt{a+x}-\sqrt{x}}=\frac{a\sqrt{a+x}(\sqrt{a+x}+\sqrt{x})}{(\sqrt{a+x})^2-(\sqrt{x})^2}$$

$$=\frac{a(a+x)+a\sqrt{ax+x^2}}{a+x-x}=\frac{a(a+x+\sqrt{ax+x^2})}{a}$$

$$=a+x+\sqrt{ax+x^2}.$$

Ex. 6. Simiplify $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)}-\dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$. [*D.B.* '28]

$$\text{The given exp.}=\frac{\sqrt{2}(2+\sqrt{3})\times\sqrt{3}(\sqrt{3}-1)}{\sqrt{3}(\sqrt{3}+1)\times\sqrt{3}(\sqrt{3}-1)}$$

$$-\frac{\sqrt{2}(2-\sqrt{3})\times\sqrt{3}(\sqrt{3}+1)}{\sqrt{3}(\sqrt{3}-1)\times\sqrt{3}(\sqrt{3}+1)}=\frac{3\sqrt{2}+\sqrt{6}}{3(3-1)}-\frac{3\sqrt{2}-\sqrt{6}}{3(3-1)}$$

$$=\frac{3\sqrt{2}+\sqrt{6}-3\sqrt{2}+\sqrt{6}}{6}=\frac{2\sqrt{6}}{6}=\frac{\sqrt{6}}{3}$$

Ex. 7. Evaluate $\sqrt{x}-\dfrac{1}{\sqrt{x}}$, when $x=3+2\sqrt{2}$. [*P. U.* '28]

$\because\ x=3+2\sqrt{2}=2+1+2\sqrt{2}=(\sqrt{2}+1)^2,\quad\therefore\ \sqrt{x}=\sqrt{2}+1.$

$$\therefore\ \sqrt{x}-\frac{1}{\sqrt{x}}=\sqrt{2}+1-\frac{1}{\sqrt{2}+1}=\sqrt{2}+1-\frac{\sqrt{2}-1}{2-1}$$

$$=\sqrt{2}+1-(\sqrt{2}-1)=\sqrt{2}+1-\sqrt{2}+1=2.$$

Ex. 8. Find the value of $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$,

when $x=\sqrt{\dfrac{n-1}{n-1}}$. [*C. U.*]

$\because\ x=\sqrt{\dfrac{n-1}{n+1}},\ \therefore\ x^2=\dfrac{n-1}{n+1},\ \therefore\ x^2-1=\dfrac{n-1}{n+1}-1=\dfrac{-2}{n+1}.$

$$\text{Now, the given exp.}=\left(\frac{x}{x-1}+\frac{x}{x+1}\right)^2-2\times\frac{x}{x-1}\times\frac{x}{x+1}$$

$$=\left(\frac{x^2+x+x^2-x}{x^2-1}\right)^2-2.\frac{x^2}{x^2-1}=\left(\frac{2x^2}{x^2-1}\right)^2-\frac{2x^2}{x^2-1}$$

$$=\left\{\frac{\frac{2(n-1)}{n+1}}{\frac{-2}{n+1}}\right\}^2-\frac{\frac{2(n-1)}{n+1}}{\frac{-2}{n+1}}=\{-(n-1)\}^2-\{-(n-1)\}$$

$$=(n-1)^2+(n-1)=(n-1)(n-1+1)=n(n-1).$$

✓**Ex. 9.** Find the value of $\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b}}}}$..........continued to infinity. [*B. U.*]

Let $x=\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b}}}}$......to infinity.

$\therefore\ x^2=a\sqrt[3]{b\sqrt{a\sqrt[3]{b}}}$......to ∞ [squaring]

$\therefore\ x^6=a^3b\sqrt{a\sqrt[3]{b}}$ ...... to $\infty=a^3bx$ [$\because$ the radical part$=x$]

$\therefore\ \frac{x^6}{x}=a^3b,$ or, $x^5=a^3b,\ \therefore\ x=\sqrt[5]{a^3b}.$

$\therefore$ The given exp.$=\sqrt[5]{a^3b}.$

Ex. 10. Simplify $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}.$ [*B. U.*]

$$\text{The exp.}=\frac{(x+\sqrt{x^2-1})^2-(x-\sqrt{x^2-1})^2}{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}=\frac{4x\sqrt{x^2-1}}{x^2-(\sqrt{x^2-1})^2}$$

$$=\frac{4x\sqrt{x^2-1}}{x^2-(x^2-1)}=\frac{4x\sqrt{x^2-1}}{x^2-x^2+1}=4x\sqrt{x^2-1}.$$

Ex. 11. If $x=\frac{\sqrt{3}}{2}$, find the value of $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ [*H. S.* '62]

$$\text{The given fraction}=\frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})}$$

$$=\frac{1+x+1-x-2\sqrt{1-x^2}}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}=\frac{2-2\sqrt{1-x^2}}{1+x-(1-x)}=\frac{2(1-\sqrt{1-x^2})}{2x}$$

$$=\frac{1-\sqrt{1-x^2}}{x}=\frac{1-\sqrt{1-\frac{3}{4}}}{\frac{\sqrt{3}}{2}}=\frac{1-\sqrt{\frac{1}{4}}}{\frac{\sqrt{3}}{2}}=\frac{1-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$=\frac{1}{\sqrt{3}}=\frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{3}}{3}.$$

Ex. 12. If $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\frac{\sqrt{3}-1}{\sqrt{3}+1}$,

find the value of $\frac{x^2+xy+y^2}{x^2-xy+y^2}$ [*H. S.* '60]

Here $x+y=\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{3-1}=\frac{8}{2}=4,$

and $xy=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}-1}{\sqrt{3}+1}=1.$

$$\therefore\quad \frac{x^2+xy+y^2}{x^2-xy+y^2}=\frac{(x+y)^2-xy}{(x+y)^2-3xy}=\frac{4^2-1}{4^2-3}=\frac{15}{13}$$

Ex. 13. If $a=\frac{\sqrt{2}+1}{\sqrt{2}-1}$ and $ab=1$, find the value of a^2+ab+b^2.

[Hints : $\because$ $ab=1$, $\because$ $b=\frac{1}{a}=\frac{\sqrt{2}-1}{\sqrt{2}+1}$.

Now proceed as in Ex. 12. Ans. 35]

Ex. 14. If $x=\frac{\sqrt{a+2b}+\sqrt{a-2b}}{\sqrt{a+2b}-\sqrt{a-2b}}$,

prove that $bx^2-ax+b=0$. [*D. B.* '44]

From the given condition we get by com. & dividendo.

$\frac{x+1}{x-1}=\frac{2\sqrt{a+2b}}{2\sqrt{a-2b}}$, $\therefore$ $\frac{(x+1)^2}{(x-1)^2}=\frac{a+2b}{a-2b}$ [squaring]

or, $\frac{x^2+2x+1}{x^2-2x+1}=\frac{a+2b}{a-2b}$, $\therefore$ $\frac{2(x^2+1)}{4x}=\frac{2a}{4b}$ [by com. & div.]

or, $\frac{x^2+1}{x}=\frac{a}{b}$, or, $bx^2+b=ax$, $\therefore$ $bx^2-ax+b=0$.

Ex. 15. Prove that $\sqrt{y+\sqrt{2xy-x^2}}+\sqrt{y-\sqrt{2xy-x^2}}=\sqrt{2x}$.

[*C. U.* '27]

$$\because \left(\sqrt{y+\sqrt{2xy-x^2}}+\sqrt{y-\sqrt{2xy-x^2}}\right)^2$$

$$=y+\sqrt{2xy-x^2}+y-\sqrt{2xy-x^2}+2\sqrt{y^2-(\sqrt{2xy-x^2})^2}$$

$$=2y+2\sqrt{y^2-2xy+x^2}=2y+2\sqrt{(x-y)^2}=2y+2(x-y)$$

$$=2y+2x-2y=2x,$$

$$\therefore \sqrt{y+\sqrt{2xy-x^2}}+\sqrt{y-\sqrt{2xy-x^2}}=\sqrt{2x}.$$

Exercise 7

1. Express $\sqrt[3]{3}$ and $\sqrt[4]{5}$ as surds of the same order.

2. Express 2, $1\frac{1}{2}$ and x^2 as surds of (i) the second order, (ii) the third order.

3. Which is greater $\sqrt[3]{5}$ or $\sqrt[4]{8}$?

4. Arrange the following in order of magnitude :—

(i) $\sqrt[3]{4}$, $\sqrt{5}$, $\sqrt[4]{12}$; (ii) 3, $\sqrt{8}$, $\sqrt[3]{10}$; (iii) $\sqrt[4]{9}$, $\sqrt[6]{25}$, $\sqrt[3]{8}$.

5. Express the following as complete surds :—

$2\sqrt{5}$, $3\sqrt[3]{2}$, $x\sqrt[4]{y}$.

6. Reduce to the simplest form ;—

(i) $\sqrt{175}$, (ii) $2\sqrt{112}$, (iii) $5\sqrt{192}$, (iv) $\sqrt[3]{375}$

(v) $4\sqrt[3]{24}-2\sqrt[3]{81}$.

7. Prove that $\sqrt{108}-\sqrt{75}=\sqrt{3}$.

8. Show that $\sqrt{98}+\sqrt{8}-2\sqrt{32}=\sqrt{2}$.

9. Find the product of the following :—

(i) $3^{\frac{1}{3}}\times\sqrt{3}$, (ii) $2\sqrt{5}\times3\sqrt{4}$, (iii) $4\sqrt[3]{4}\times2\sqrt[3]{2}$
(iv) $2\sqrt[3]{3}\times3\sqrt{6}$.

10. Divide :—

(i) $6\sqrt{5}$ by $2\sqrt{2}$, (ii) $4\sqrt[3]{4}$ by $2\sqrt[3]{2}$, (iii) $3\sqrt[4]{5}\div6\sqrt[4]{4}$.

11. If $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$ and $\sqrt{5}=2{\cdot}236$, find the value of the following correct to 2 places of decimals :—

(i) $\dfrac{3}{2\sqrt{2}}$, (ii) $\dfrac{4}{\sqrt{3}}$, (iii) $\dfrac{15}{\sqrt{5}}$ (iv) $\dfrac{1}{\sqrt{3}-\sqrt{2}}$, (v) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$.

12. Multiply :—(i) $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$,
(ii) $\sqrt{3}+\sqrt{7}$ by $\sqrt{2}+\sqrt{3}$, (iii) $2\sqrt{3}+3\sqrt{2}$ by $4\sqrt{2}+\sqrt{5}$,
(iv) $3\sqrt{5}-2\sqrt{6}$ by $2\sqrt{3}-\sqrt{5}$, (v) $\sqrt{x}+\sqrt{y-x}$ by $\sqrt{x}-\sqrt{y-x}$,
(vi) $\sqrt{a+b}-\sqrt{a-b}$ by $\sqrt{a-b}$,
(vii) $\sqrt{3}+\sqrt{5}+\sqrt{6}$ by $\sqrt{3}+\sqrt{6}-\sqrt{5}$, (viii) $x+\sqrt{y}$ by $\sqrt{x-y}$.

13. Find the square of :—
(i) $\sqrt{3}-\sqrt{2}$, (ii) $2-\sqrt{3}$ (iii) $\sqrt{2}+2\sqrt{3}$,
(iv) $\sqrt{x+y}-\sqrt{x-y}$, (v) $a-\sqrt{b^2-a^2}$, (vi) $\sqrt{2x+5}+\sqrt{2x-5}$.

14. Rationalise the denominators :—
(a) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$, (b) $\frac{3\sqrt{2}+1}{2\sqrt{5}+1}$ (c) $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$, (d) $\frac{a\sqrt{a+x}}{\sqrt{a+x}-\sqrt{x}}$ [*B. U.*]

15. Divided $3+\sqrt{6}$ by $\sqrt{3}+\sqrt{2}$.

16. Find the value of (a) $\frac{2\sqrt{5}+1}{\sqrt{5}+1}$, and (b) $\frac{8+3\sqrt{2}}{3+\sqrt{5}}+\frac{8-3\sqrt{2}}{3-\sqrt{5}}$.

Simplify :

17. $\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}$

18. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$. [*B. U.*]

19. $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$. [*B. U.* 1888]

20. $\frac{10\sqrt{2}}{\sqrt{18}-\sqrt{(3+\sqrt{5})}}-\frac{\sqrt{10}+\sqrt{18}}{\sqrt{18}-\sqrt{(3-\sqrt{5})}}$. [*M. U.* 1892]

21. $\frac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}-\sqrt{a+x}}-\frac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}+\sqrt{a+x}}$. [*M. U.*]

22. Evaluate $\sqrt{x\sqrt{x\sqrt{x}}}\cdots$ to infinity.

23. If $a=\frac{\sqrt{5}+1}{\sqrt{5}-1}$, $b=\frac{\sqrt{5}-1}{\sqrt{5}+1}$, find the value of $\frac{a^2+ab+b^2}{a^2-ab+b^2}$.

23. (*a*) It $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $xy=1$, find the value of x^2+xy+y^2.

24. Find the value of $\sqrt{x}+\dfrac{1}{\sqrt{x}}$, when $x=4+2\sqrt{3}$.

25. If $x=\dfrac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}$,

show that $b^2x^2-2a^2x+b^2=0$.

26. Prove that $\sqrt{a^2+2x\sqrt{a^2-x^2}}+\sqrt{a^2-2x\sqrt{a^2-x^2}}=2x$.

27. Evaluate $a^6+a^4+a^2+1$, when $a=\dfrac{1+\sqrt{-1}}{\sqrt{2}}$.

$$\left[\text{Hints}: \ a^2=\frac{1-1+2\sqrt{-1}}{2}=\sqrt{-1}, \quad \therefore \ a^4=-1\right]$$

28. If $x=(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}$, find the value of $x^3+3bx-2a$. [*H. S.* '65]

[Hints: $x^3=a+\sqrt{a^2+b^3}+a-\sqrt{a^2+b^3}+3(a+\sqrt{a^2+b^3})^{\frac{1}{3}}\times$

$$(a-\sqrt{a^2+b^3})^{\frac{1}{3}}\{(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}\}$$

$$=2a+3\{a^2-(\sqrt{a^2+b^3})^2\}^{\frac{1}{3}}x=2a+3(a^2-a^2-b^3)^{\frac{1}{3}}\cdot x$$

$$=2a+3(-b^3)^{\frac{1}{3}}\cdot x=2a+3.(-1)^{\frac{1}{3}}.(b^3)^{\frac{1}{3}}.x=2a-3bx\cdots]$$

29. If $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, then $(x+y+z)^3=27xyz$.

30. If $x=a^{\frac{1}{3}}b^{-\frac{1}{3}}+a^{-\frac{1}{3}}b^{\frac{1}{3}}$, prove that $a(bx^3-3bx-a)=b^2$.

[*C. U. (D. M. H.)* '51]

[Hints: $x^3=ab^{-1}+a^{-1}b+3a^{\frac{1}{3}}\cdot b^{-\frac{1}{3}}\cdot a^{-\frac{1}{3}}\cdot b^{\frac{1}{3}}\left(a^{\frac{1}{3}}b^{-\frac{1}{3}}+a^{-\frac{1}{3}}b^{\frac{1}{3}}\right)$

$$=\frac{a}{b}+\frac{b}{a}+3a^0b^0\cdot x=\frac{a}{b}+\frac{b}{a}+3x$$

$\therefore$ $abx^3=a^2+b^2+3abx$ (multiplying both sides by ab),

or, $abx^3-3abx-a^2=b^2$, $\therefore$ $a(bx^3-3bx-a)=b^2$.]

INSTITUTE OF EDUCATION FOR WOMEN
2659
Dept. of Extension
SERVICE
20.3.69
CALCUTTA-27

EVOLUTION

We have already dealt with **Involution**, which is the process of raising a given expression to any required power. Here we shall deal with **Evolution**.

39. Root or Base : Any root of a given expression is that quantity which will produce the given expression when raised to the **power** denoted by the index of the root.

Thus $\sqrt[3]{8}=2$ (here, 2 raised to the third power produces 8). $\therefore$ 2 is the cube root of 8.

Again, $\sqrt{a^2+2ab+b^2}=a+b$ (here, $a+b$ raised to the second power produces a^2+b^2+2ab), $\therefore$ $a+b$ is the square root of $a^2+2ab+b^2$.

The symbols $\sqrt{\ }$ or $\sqrt[3]{\ }$ etc. are called radical signs.

40. Evolution : The process of finding any proposed root of a given expression is called **evolution.** Thus evolution is the reverse of involution.

By this process we find the square root or cube root, or nth. root etc. of a given expression.

Ex. 1. $\sqrt{a^4}=\pm a^2 \quad [\because \ (\pm a^2)^2=a^4]$

$\sqrt{a^5}=\pm a^{\frac{5}{2}} \quad [\because \ (\pm a^{\frac{5}{2}})^2=a^5]$

$\sqrt{16a^6}=\pm 4a^3$.

N. B. From the above examples we find that

(i) a positive quantity has two square roots, one positive and the other negative.

Since $(a^2)^2=a^4$ and $(-a^2)^2=a^4$, the square root of a^4 is $+a^2$ and $-a^2$. It is indicated by the double sign ±. Thus $\sqrt{a^4}=\pm a^2$.

(*ii*) The index of the square root of any factor is half the given index of the factor. Thus $\sqrt{a^5}=a^{\frac{5}{2}}$, $\sqrt{a^n}=a^{\frac{n}{2}}$.

(*iii*) the square root of the numerical coefficient of the given quantity is found by Arithmetic.

Ex. 2. $\sqrt[3]{x^6}=x^2$ [for $(x^2)^3=x^6$]

$\sqrt[3]{-x^6}=-x^2$ [for $(-x^2)^3=-x^6$]

$\sqrt[3]{27x^3y^6}=3xy^2$.

N. B. From the above examples we find that

(*a*) the index of the cube root of a quantity is one-third of the index of the given quantity. Thus, the cube root of x^6 is x^2, the cube root of x^5 is $x^{\frac{5}{3}}$ etc.

Rule. In the extraction of any root of a given expression the index of each literal factor is found by dividing the index in the expression by the index of the root required.

Thus, $\sqrt[4]{a^{20}}=\pm a^{\frac{20}{4}}=\pm a^5$, $\sqrt[5]{-x^{15}}=-x^{\frac{15}{5}}=-x^3$,

$\sqrt[3]{x^6y^n}=x^2y^{\frac{n}{3}}$.

Hence from the **Rule of signs**, we know that (i) any **even root** of a positive quantity will have the double sign ± (*i.e.*, the roots are equal in magnitude but opposite in sign).

(*b*) Any **odd root** of a quantity has the same sign as the given quantity (*i. e.*, any odd root of a positive quantity is positive and that of a negative quantity is negative).

(*c*) A negative quantity can have no even root.

Thus $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-x}$ etc. have no arithmetical significance. Such quantities are called **unreal** or **imaginary**.

SQUARE ROOT

41. Square root : The square root of any quantity (number or expression) is a quantity which, being multiplied by itself, produces the given quantity.

Thus $3\times3=9$, so 3 is the square root of 9 ; $a\times a=a^2$, so a is the square root of a^2 ; $(x+y)\times(x+y)=x^2+2xy+y^2$, so $(x+y)$ is the square root of $x^2+2xy+y^2$.

You have learnt above how to determine the index of the square root of each literal factor in the given expression.

42. (*i*) Extraction of square root by inspection :

If by looking into the form of the terms of the given expression, it is found to be a perfect square, its root can at once be found by inspection.

Example. Extract the square root of $9x^2-24xy+16y^2$.

The given exp. $=(3x)^2-2.3x.4y+(4y)^2=(3x-4y)^2$

$\therefore$ The required square root $=3x-4y$.

[**N. B.** Each expression has two square roots, equal in magnitude but opposite in sign. Hence the square root of the above expression is $\pm\ (3x-4y)$. We, however, generally show the positive square root only. The students should give both the roots in such cases, as shown here.]

(*ii*) General process of finding the square root.

Rule : First arrange the given expression in the descending or ascending powers of any of the letters (say a or x) involved. Then find the square root of the first term and put it as the first term of the square root, and subtract its square from the given expression. Now multiply this first term of the root by 2 and put it as the first term of the divisor. Find the second term of the square root by dividing the first term of the remainder by the first term of the divisor, and put it as the second term of the square root as well as of the divisor. Now multiply the whole divisor by the second term and subtract the product from the first remainder.

(*ii*) The index of the square root of any factor is half the given index of the factor. Thus $\sqrt{a^5}=a^{\frac{5}{2}}$, $\sqrt{a^n}=a^{\frac{n}{2}}$.

(*iii*) the square root of the numerical coefficient of the given quantity is found by Arithmetic.

Ex. 2. $\sqrt[3]{x^6}=x^2$ [for $(x^2)^3=x^6$]

$\sqrt[3]{-x^6}=-x^2$ [for $(-x^2)^3=-x^6$]

$\sqrt[3]{27x^3y^6}=3xy^2$.

N. B. From the above examples we find that

(*a*) the index of the cube root of a quantity is one-third of the index of the given quantity. Thus, the cube root of x^6 is x^2, the cube root of x^5 is $x^{\frac{5}{3}}$ etc.

Rule. In the extraction of any root of a given expression the index of each literal factor is found by dividing the index in the expression by the index of the root required.

Thus, $\sqrt[4]{a^{20}}=\pm a^{\frac{20}{4}}=\pm a^5$, $\sqrt[5]{-x^{15}}=-x^{\frac{15}{5}}=-x^3$,

$$\sqrt[3]{x^6y^n}=x^2y^{\frac{n}{3}}.$$

Hence from the **Rule of signs**, we know that (i) any **even root** of a positive quantity will have the double sign ± (*i.e.*, the roots are equal in magnitude but opposite in sign).

(*b*) Any **odd root** of a quantity has the same sign as the given quantity (*i. e.*, any odd root of a positive quantity is positive and that of a negative quantity is negative).

(*c*) A negative quantity can have no even root.

Thus $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-x}$ etc. have no arithmetical significance. Such quantities are called **unreal or imaginary.**

SQUARE ROOT

41. Square root : The square root of any quantity (number or expression) is a quantity which, being multiplied by itself, produces the given quantity.

Thus $3\times3=9$, so 3 is the square root of 9 ; $a\times a=a^2$, so a is the square root of a^2 ; $(x+y)\times(x+y)=x^2+2xy+y^2$, so $(x+y)$ is the square root of $x^2+2xy+y^2$.

You have learnt above how to determine the index of the square root of each literal factor in the given expression.

42. (*i*) Extraction of square root by inspection :

If by looking into the form of the terms of the given expression, it is found to be a perfect square, its root can at once be found by inspection.

Example. Extract the square root of $9x^2-24xy+16y^2$.

The given exp. $=(3x)^2-2.3x.4y+(4y)^2=(3x-4y)^2$

$\therefore$ The required square root $=3x-4y$.

[**N. B.** Each expression has two square roots, equal in magnitude but opposite in sign. Hence the square root of the above expression is $\pm\ (3x-4y)$. We, however, generally show the positive square root only. The students should give both the roots in such cases, as shown here.]

(*ii*) General process of finding the square root.

Rule : First arrange the given expression in the descending or ascending powers of any of the letters (say a or x) involved. Then find the square root of the first term and put it as the first term of the square root, and subtract its square from the given expression. Now multiply this first term of the root by 2 and put it as the first term of the divisor. Find the second term of the square root by dividing the first term of the remainder by the first term of the divisor, and put it as the second term of the square root as well as of the divisor. Now multiply the whole divisor by the second term and subtract the product from the first remainder.

Then obtain the new divisor (*i.e*, the second divisor) by doubling the first two terms of the square root. Then we find the third term of the square root by dividing the first term of the second remainder by the first term of the new divisor.

This process is extended if the expression is multinomial.

The full process is shown in the following examples.

Examples 10

Ex. 1. Find the square root of $x^4+4x^3+10x^2+12x+9$. [*C. U.* '22]

$$
\begin{array}{r|l|l}
 & x^4+4x^3+10x^2+12x+9 & x^2+2x+3 \\
 & x^4 & \\ \hline
2x^2+2x & 4x^3+10x^2+12x+9 & \\
 & 4x^3+4x^2 & \\ \hline
2x^2+4x+3 & 6x^2+12x+9 & \\
 & 6x^2+12x+9 & \\ \hline
\end{array}
$$

$\therefore$ The required square root $= \pm(x^2+2x+3)$.

Ex. 2. Extract the square root of $25x^{-2}-12x+16x^{-8}+4x^4-24x^{-5}$. [*C. U.* '12]

$$
\begin{array}{r|l|l}
 & 4x^4-12x+25x^{-2}-24x^{-5}+16x^{-8} & 2x^2-3x^{-1}+4x^{-4} \\
 & 4x^4 & \\ \hline
4x^2-3x^{-1} & -12x+25x^{-2}-24x^{-5}+16x^{-8} & \\
 & -12x+\ 9x^{-2} & \\ \hline
4x^2-6x^{-1}+4x^{-4} & 16x^{-2}-24x^{-5}+16x^{-8} & \\
 & 16x^{-2}-24x^{-5}+16x^{-8} & \\ \hline
\end{array}
$$

$\therefore$ The required square root $= \pm(2x^2-3x^{-1}+4x^{-4})$.

[**N. B.** The expression is arranged in the descending powers of x. The power of x^{-2} is less than the power of x, for in x the index of the power is 1 and in x^{-2} the index is -2 which is less than 1. Here note that to find the second term of the root, $-12x$ has to be divided by $4x^2$. By dividing -12 by 4 we get -3 and as in division the indices of the powers are subtracted $x \div x^2 = x^{1-2} = x^{-1}$. $\therefore$ The second term is $-3x^{-1}$. Similarly, dividing $16x^{-2}$ by $4x^2$ we have $4x^{-2-2}$ or $4x^{-4}$ as the third term of the root.]

Ex. 3. Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}-6\frac{x}{y}+6\frac{y}{x}+7$. [*C.U.*'11]

$$\begin{array}{l|l|l}
 & \frac{x^2}{y^2}-\frac{6x}{y}+7+\frac{6y}{x}+\frac{y^2}{x^2} & \frac{x}{y}-3-\frac{y}{x} \\
 & \frac{x^2}{y^2} & \\
\hline
\frac{2x}{y}-3 & -\frac{6x}{y}+7+\frac{6y}{x}+\frac{y^2}{x^2} & \\
 & -\frac{6x}{y}+9 & \\
\hline
\frac{2x}{y}-6-\frac{y}{x} & -2+\frac{6y}{x}+\frac{y^2}{x^2} & \\
 & -2+\frac{6y}{x}+\frac{y^2}{x^2} & \\
\hline
\end{array}$$

$\therefore$ the required square root $=\pm\left(\frac{x}{y}-3-\frac{y}{x}\right)$

[**N. B.** Here $6\frac{x}{y}=\frac{6x}{y}$. In arranging the given exp. in the descending powers of x, we put $\frac{x^2}{y^2}$ as the first term and $-\frac{6x}{y}$ as the second term. Then we put 7 as the third term, because 7 is taken as $7x^0$. Next we put $\frac{6y}{x}$ as the fourth term and $\frac{y^2}{x^2}$ as the fifth term, for $\frac{6y}{x}=6yx^{-1}$ and $\frac{y^2}{x^2}=y^2x^{-2}$, Hence, the successive indices of the powers of x are 2, 1, 0, -1, -2.]

Ex. 4. Extract the square root of

$$9x^{\frac{3}{2}}+6+4x-12x^{\frac{5}{4}}+x^{-\frac{3}{2}}-3x^{-\frac{1}{4}}.$$

$$\begin{array}{l|l|l}
 & 9x^{\frac{3}{2}}-12x^{\frac{5}{4}}+4x+6-4x^{-\frac{1}{4}}+x^{-\frac{3}{2}} & 3x^{\frac{3}{4}}-2x^{\frac{1}{2}}+x^{-\frac{3}{4}} \\
 & 9x^{\frac{3}{2}} & \\
\hline
6x^{\frac{3}{4}}-2x^{\frac{1}{2}} & -12x^{\frac{5}{4}}+4x+6-4x^{-\frac{1}{4}}+x^{-\frac{3}{2}} & \\
 & -12x^{\frac{5}{4}}+4x & \\
\hline
6x^{\frac{3}{4}}-4x^{\frac{1}{2}}+x^{-\frac{3}{4}} & 6-4x^{-\frac{1}{4}}+x^{-\frac{3}{2}} & \\
 & 6-4x^{-\frac{1}{4}}+x^{-\frac{3}{2}} & \\
\hline
\end{array}$$

$\therefore$ The required square root $=\pm\left(3x^{\frac{3}{4}}-2x^{\frac{1}{2}}+x^{-\frac{3}{4}}\right)$.

Ex. 5. Find the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$. [*C. U.* '51]

$$
\begin{array}{r|l|l}
 & x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16} & x^2-x+\frac{1}{4} \\
 & x^4 & \\ \hline
2x^2-x & -2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16} & \\
 & -2x^3+x^2 & \\ \hline
2x^2-2x+\frac{1}{4} & \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} & \\
 & \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} & \\ \hline
\end{array}
$$

$\therefore$ The required square root $=\pm(x^2-x+\frac{1}{4})$.

Ex. 6. Find the square root of $4x^4+20x^2-3-\frac{70}{x^2}+\frac{49}{x^4}$ [*C. U.* '10, '43]

$$
\begin{array}{r|l|l}
 & 4x^4+20x^2-3-\frac{70}{x^2}+\frac{49}{x^4} & 2x^2+5-\frac{7}{x^2} \\
 & 4x^4 & \\ \hline
4x^2+5 & 20x^2-3-\frac{70}{x^2}+\frac{49}{x^4} & \\
 & 20x^2+25 & \\ \hline
4x^2+10-\frac{7}{x^2} & -28-\frac{70}{x^2}+\frac{49}{x^4} & \\
 & -28-\frac{70}{x^2}+\frac{49}{x^4} & \\ \hline
\end{array}
$$

$\therefore$ The required square root $=\pm\left(2x^2+5-\frac{7}{x^2}\right)$.

Ex. 7. Extract the sq. root of $9a-12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1}$. [*Pat. U.* '30]

$$
\begin{array}{r|l|l}
 & 9a-12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1} & 3a^{\frac{1}{2}}-2-a^{-\frac{1}{2}} \\
 & 9a & \\ \hline
6a^{\frac{1}{2}}-2 & -12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1} & \\
 & -12a^{\frac{1}{2}}+4 & \\ \hline
6a^{\frac{1}{2}}-4-a^{-\frac{1}{2}} & -6+4a^{-\frac{1}{2}}+a^{-1} & \\
 & -6+4a^{-\frac{1}{2}}+a^{-1} & \\ \hline
\end{array}
$$

$\therefore$ The required square root. $=\pm\left(3a^{\frac{1}{2}}-2-a^{-\frac{1}{2}}\right)$.

[**N. B.** The square root of a is $a^{\frac{1}{2}}$. In finding the third term of the square root, -6 is divided by $6a^{\frac{1}{2}}$.

Now $-6\div 6a^{\frac{1}{2}}=\dfrac{-6}{6a^{\frac{1}{2}}}=-\dfrac{1}{a^{\frac{1}{2}}}=-a^{-\frac{1}{2}}$;

Otherwise, $-6\div 6a^{\frac{1}{2}}=-6a^{0}\div 6a^{\frac{1}{2}}=-a^{0-\frac{1}{2}}=-a^{-\frac{1}{2}}$.]

Ex. 8. Find the sq. root of $\left(x^4+\dfrac{1}{x^4}\right)+6\left(x+\dfrac{1}{x}\right)\left(x-\dfrac{1}{x}\right)+7$.

[*C. U.* '47]

The given exp. $=x^4+\dfrac{1}{x^4}+6\left(x^2-\dfrac{1}{x^2}\right)+7=x^4+\dfrac{1}{x^4}+6x^2-\dfrac{6}{x^2}+7$.

$$\begin{array}{r l}
 & x^4+6x^2+7-\frac{6}{x^2}+\frac{1}{x^4}\;\Big(\,x^2+3-\frac{1}{x^2} \\
 & x^4 \\ \hline
2x^2+3\;\big) & 6x^2+7-\frac{6}{x^2}+\frac{1}{x^4} \\
 & 6x^2+9 \\ \hline
2x^2+6-\frac{1}{x^2}\;\big) & -2-\frac{6}{x^2}+\frac{1}{x^4} \\
 & -2-\frac{6}{x^2}+\frac{1}{x^4} \\ \hline
\end{array}$$

$\therefore$ The required square root $=\pm\left(x^2+3-\dfrac{1}{x^2}\right)$.

[**N. B.** This can be worked out as in Ex. 10.]

Ex. 9. Find the first three terms of the square root of a^2+b^2.

$$\begin{array}{r l}
 & a^2+b^2\;\Big(\;a+\frac{b^2}{2a}-\frac{b^4}{8a^3} \\
 & a^2 \\ \hline
2a+\frac{b^2}{2a}\;\big) & b^2 \\
 & b^3+\frac{b^4}{4a^2} \\ \hline
2a+\frac{b^2}{a}-\frac{b^4}{8a^3}\;\big) & -\frac{b^4}{4a^2} \\
 & -\frac{b^4}{4a^2}-\frac{b^6}{8a^4}+\frac{b^8}{64a^6} \\ \hline
 & \frac{b^6}{8a^4}-\frac{b^8}{64a^6}
\end{array}$$

$\therefore$ The first three terms of the square root $=a+\dfrac{b^2}{2a}-\dfrac{b^4}{8a^3}$.

43. We may find the square root by reducing the given expression into the form of a complete square.

Ex. 10. Find the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.

[*C. U.* '14 ; *D. B.* '38 ; *E. B. S. B.* '50 ; *G. U.* '48 ; *P. U.* '27]

The given exp. $=\left(x-\frac{1}{x}\right)^2+4x\cdot\frac{1}{x}-4\left(x-\frac{1}{x}\right)$

$[\because \ (a+b)^2=(a-b)^2+4ab]$

$=\left(x-\frac{1}{x}\right)^2+4-2.2\left(x-\frac{1}{x}\right)$

$=a^2-4a+4$ $\left[\text{putting } a \text{ for } x-\frac{1}{x}\right]=(a-2)^2=\left(x-\frac{1}{x}-2\right)^2$.

$\therefore$ The required square root $=\pm\left(x-2-\frac{1}{x}\right)$.

[**N. B.** To reduce an expression into the form of a perfect square, we have to express it in the form of $(a)^2\pm2ab+(b)^2$, *i.e.*, the sum of the squares of two quantities together with (or diminished by) twice their product. In the above example, $-4\left(x-\frac{1}{x}\right)$ is not a perfect square, so it must be twice the product of two terms of the square root. $\therefore$ One term must be $\left(x-\frac{1}{x}\right)^2$.

Hence $\left(x+\frac{1}{x}\right)^2$ is shown as $\left(x-\frac{1}{x}\right)^2+4.x.\frac{1}{x}$.]

Ex. 11. Extract the square root of $4\left(x^2+\frac{1}{x^2}\right)-12\left(x-\frac{1}{x}\right)+1$,

[*B. U.*]

The given exp. $=4\left\{\left(x-\frac{1}{x}\right)^2+2x.\frac{1}{x}\right\}-12\left(x-\frac{1}{x}\right)+1$

$=4(a^2+2)-12a+1$ $\left[\text{putting } a \text{ for } x-\frac{1}{x}\right]=4a^2-12a+9$

$=(2a-3)^2=\left(2x-\frac{2}{x}-3\right)^2$ [putting the value of a]

$\therefore$ The required square root $=\pm\left(2x-3-\frac{2}{x}\right)$.

Ex. 12. Find the square root of $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3$.

[*C. U.* '40]

$$\text{The given exp.}=\left(x^2+\frac{1}{x^2}\right)^2-2x^2.\frac{1}{x^2}+2\left(x^2+\frac{1}{x^2}\right)+3$$

$$=\left(x^2+\frac{1}{x^2}\right)^2+2\left(x^2+\frac{1}{x^2}\right)+1=\left(x^2+\frac{1}{x^2}+1\right)^2$$

$\therefore$ The required square root $=\pm\left(x^2+1+\frac{1}{x^2}\right)$.

Ex. 13. Find the square root of $\left(a+\frac{1}{2a}\right)^2-14\left(a-\frac{1}{2a}\right)+47$.

[*C. U.* '19 ; *W. B. S. F.* '58]

$$\text{The given exp.}=\left(a-\frac{1}{2a}\right)^2+4.a.\frac{1}{2a}-14\left(a-\frac{1}{2a}\right)+47$$

$$=\left(a-\frac{1}{2a}\right)^2+2-14\left(a-\frac{1}{2a}\right)+47=\left(a-\frac{1}{2a}\right)^2-2.\ 7.\left(a-\frac{1}{2a}\right)+49$$

$$=\left(a-\frac{1}{2a}-7\right)^2.\quad\therefore\ \text{The reqd. sq. root}=\pm\left(a-7-\frac{1}{2a}\right)$$

Ex. 14. Extract the square root of

$$\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^3}+4\frac{a}{a+b}\times\frac{b}{a-b}.$$

[*C. U.* ; *B. U.*]

$$\text{The given exp.}=\frac{(a^2+b^2)^2}{(a^2-b^2)^2}+4\frac{ab}{a^2-b^2}=\frac{(a^2-b^2)^2+4a^2b^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}$$

$$=\frac{(a^2-b^2)^2}{(a^2-b^2)^2}+\frac{4a^2b^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}$$

$$=1+\left(\frac{2ab}{a^2-b^2}\right)^2+2.\ \frac{2ab}{a^2-b^2}=\left(1+\frac{2ab}{a^2-b^2}\right)^2$$

$\therefore$ the required square root $=\pm\left(1+\frac{2ab}{a^2-b^2}\right)$.

Ex. 15. Find the sq. root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\frac{1}{2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}$.

The given exp. $=\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\right)+\frac{1}{2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}$

$$=\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2-2.\ \frac{1}{\sqrt{2}}\left(\frac{x}{y}+\frac{y}{x}\right)=\left(\frac{x}{y}+\frac{y}{x}-\frac{1}{\sqrt{2}}\right)^2$$

$\therefore$ The required square root $=\pm\left(\frac{x}{y}-\frac{1}{\sqrt{2}}+\frac{y}{x}\right)$.

[N. B. $2\frac{1}{2}=2+\frac{1}{2}$; taking 2 with $\frac{x^2}{y^2}+\frac{y^2}{x^2}$ we have $\frac{x^2}{y^2}+\frac{y^2}{x^2}+2$ $=\left(\frac{x}{y}+\frac{y}{x}\right)^2$; $\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2$ and $\sqrt{2}=2\times\frac{1}{\sqrt{2}}$.]

Ex. 16. Show that $(x-1)(x-3)(x-5)(x-7)+16$ is a perfect square. [*C. U.* '30, '36, '41]

The given exp. $=\{(x-1)(x-7)\}\{(x-3)(x-5)\}+16$

$=(x^2-8x+7)(x^2-8x+15)+16$

$=(a+7)(a+15)+16$ [putting a for x^2-8x]

$=a^2+22a+105+16=a^2+22a+121=(a+11)^2$

$=(x^2-8+11)^2$, which is a perfect square.

Ex. 17. Find the square root of $x(x+1)(x+2)(x+3)+1$. [*G. U.* '51 ; *C. U.* '24]

The exp. $=x(x+3)(x+1)(x+2)+1=(x^2+3x)(x^2+3x+2)+1$ $=a(a+2)+1$ [putting a for x^2+3x] $=a^2+2a+1=(a+1)^2$ $=(x^2+3x+1)^2$. $\therefore$ The required square root $=\pm(x^2+3x+1)$.

Ex. 18. Express $(x-2a)(x-5a)(x-8a)(x-11a)+81a^4$ as a perfect square. [*C. U.* '45]

The given exp. $=(x-2a)(x-11a)(x-5a)(x-8a)+81a^4$

$=(x^2-13ax+22a^2)+(x^2-13ax+40a^2)+81a^4$

$=(k+22a^2)(k+40a^2)+81a^4$ [putting k for x^2-13ax]

$=k^2+62ka^2+880a^4+81a^4=k^2+62ka^2+961a^4=(k+31a^2)^2$

$=(x^2-13ax+31a^2)^2$, which is a perfect square.

Ex. 19. Find the square root of $(ab+ac+bc)^2-4abc(a+c)$.

Let $ab+bc=x$, and $ac=y$, then $x=b(a+c)$. and $xy=abc(a+c)$.

Now, the given $\text{exp.}=(x+y)^2-4xy=(x-y)^2=(ab+bc-ac)^2$.

$\therefore$ The required square root $=\pm(ab+bc-ac)$.

Ex. 20. Find the square root of $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$. [*C. U.* '11]

$$\text{The given exp.}=\{(a-b)^2\}^2-2(a-b)^2(a^2+b^2)+(a^2+b^2)^2+a^4+b^4-2a^2b^2 \quad [\because\ 2(a^4+b^4)=(a^4+b^4)+a^4+b^4=(a^2+b^2)^2-2a^2b^2+a^4+b^4]$$

$$=\{(a-b)^2-(a^2+b^2)\}^2+a^4+b^4-2a^2b^2$$
$$=\{a^2+b^2-2ab-a^2-b^2\}^2+a^4+b^4-2a^2b^2$$
$$=(-2ab)^2+a^4+b^4-2a^2b^2=4a^2b^2+a^4+b^4-2a^2b^2$$
$$=a^4+b^4+2a^2b^2=(a^2+b^2)^2.$$

$\therefore$ The required square root $=\pm(a^2+b^2)$.

Ex. 21. Extract the sq. root of $(1-6a^2+a^4)^2+16a^2(1-a^2)^2$. [*C. U.* '20]

$$\text{The given exp.}=\{(1-2a^2+a^4)-4a^2\}^2+16a^2(1-a^2)^2$$
$$=\{(1-a^2)^2-4a^2\}^2+16a^2(1-a^2)^2$$
$$=\{(1-a^2)^2\}^2+16a^4-8a^2(1-a^2)^2+16a^2(1-a^2)^2$$
$$=\{(1-a^2)^2\}^2+16a^4+8a^2(1-a^2)^2$$
$$=\{(1-a^2)^2\}^2+(4a^2)^2+2\ 4a^2(1-a^2)^2=\{(1-a^2)^2+4a^2\}^2$$
$$=(1+a^4-2a^2+4a^2)^2=(1+2a^2+a^4)^2$$

$\therefore$ The required square root $=\pm(1+2a^2+a^4)$.

44. Square root of Quadratic Surds :

Let us here discuss some properties of Quadratic surds.

Theorem 1. A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.

Proof : If possible let $\sqrt{x}=a\pm\sqrt{y}$.

By squaring we have $x=a^2+y\pm 2a\sqrt{y}$,

or, $\pm 2a\sqrt{y}=x-a^2-y$, or, $\pm\sqrt{y}=\frac{x-a^2-y}{2a}$ (which is rational).

Thus here a surd is equal to a rational quantity, which is absurd.

Hence, the supposition being absurd, the theorem is proved.

Theorem 2. If $x+\sqrt{y}=a+\sqrt{b}$, where x and a are both rational and $\sqrt{y}$ and $\sqrt{b}$ are both irrational, then $x=a$ and $y=b$.

Proof : If x and a are not equal, let $x=a+m$.

Now, from the given condition, we have $x+\sqrt{y}=a+\sqrt{b}$,

or, $a+m+\sqrt{y}=a+\sqrt{b}$, $\therefore\ \sqrt{b}=m+\sqrt{y}$,

i.e. a quadratic surd is equal to the sum of a rational quantity and a surd, which is absurd.

$\therefore\ x=a$. Hence $\sqrt{y}=\sqrt{b}$, $\therefore\ y=b$.

[N. B. (i) The above theorem will not hold good, if $\sqrt{y}$ and $\sqrt{b}$ are not really irrational. Thus $3+\sqrt{16}=4+\sqrt{9}$ (both sides being equal to 7), but we cannot deduce here that $3=4$ and $\sqrt{16}=\sqrt{9}$. As in this example $\sqrt{16}$ and $\sqrt{9}$ are not real surds, the theorem does not hold good.

(ii) By the above proposition, if $x+\sqrt{y}=a+\sqrt{b}$, then $x-\sqrt{y}=a-\sqrt{b}$.]

(1) To find the square root by **inspection.**

First express the binomial quadratic surd in the form $a+2\sqrt{b}$ and then find by inspection two quantities, whose sum is a and product is b.

Examples 11

Ex. 1. Find the square root of $7+2\sqrt{12}$.

Here we have to find two numbes whose sum is 7 and product is 12. Evidently 4 and 3 are the numbers.

Now the given exp. $=4+3+2\sqrt{4.3}$

$$=(2)^2+(\sqrt{3})^2+2.2.\sqrt{3}=(2+\sqrt{3})^2$$

$\therefore$ The required square root $=\pm(2+\sqrt{3})$.

[N. B. Since every quantity has two square roots, equal in magnitude but opposite in sign, here the other square root is $-2-\sqrt{3}$. $\therefore$ The requare root is $\pm(2+\sqrt{3})$. Generally only the positive root is shown. The students should show both the roots.]

Ex. 2. Extract the square root of $28-10\sqrt{3}$.

The given exp. $=28-2.5\sqrt{3}=25+3-2.5\sqrt{3}$

$$=(5)^2+(\sqrt{3})^2-2.5.\sqrt{3}=(5-\sqrt{3})^2,$$

$\therefore$ The required square root $=\pm(5-\sqrt{3})$.

Ex. 3. Find the square root of $8+2\sqrt{15}$.

The given exp. $=5+3+2\sqrt{5.3}$

$$=(\sqrt{5})^2+(\sqrt{3})^2+2.\sqrt{5}.\sqrt{3}=(\sqrt{5}+\sqrt{3})^2.$$

$\therefore$ The required square root $=\pm(\sqrt{5}+\sqrt{3})$.

Ex. 4. Find the square root of $2-\sqrt{3}$.

$$2-\sqrt{3}=2-2.\tfrac{1}{2}.\sqrt{3}=2-2\sqrt{\tfrac{1}{4}}.\sqrt{3}=2-2\sqrt{\tfrac{3}{4}}$$

$$=\tfrac{3}{2}+\tfrac{1}{2}-2\sqrt{\tfrac{3}{2}.\tfrac{1}{2}}=(\sqrt{\tfrac{3}{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2-2.\sqrt{\tfrac{3}{2}}.\sqrt{\tfrac{1}{2}}.$$

$=\left(\sqrt{\frac{3}{2}}-\frac{1}{\sqrt{2}}\right)^2$. $\therefore$ The required square root $=\pm(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})$.

(2) In equations of the from $x\pm\sqrt{y}=m\pm\sqrt{n}$, the rational parts of the two sides are equal and their irrational parts are equal. This rule may be applied in finding the square root.

Ex. 5. Find the square root of $8-2\sqrt{15}$.

$$\text{Let } \sqrt{8-21\sqrt{5}}=\sqrt{x}-\sqrt{y}$$

Squaring we have, $8-2\sqrt{15}=x+y-2\sqrt{xy}$

Now, equating the rational parts and the irrational parts we have $x+y=8$ and $-2\sqrt{xy}=-2\sqrt{15}$, *i.e.*, $xy=15$.

$\therefore\ (x+y)^2=64$ and $4xy=4.15=60$.

$\therefore\ (x-y)^2=(x+y)^2-4xy=64-60=4, \quad \therefore\ x-y=\pm 2$.

Now, $x+y=8\ldots(1)$, and $x-y=\pm 2\ldots(2)$,

Solving (1) and (2) we have

$$x=5,\ y=3, \text{ and } x=3,\ y=5.$$

$\therefore$ The required square root $=\pm(\sqrt{5}-\sqrt{3})$.

Ex. 6. Find the square root of $a+b+\sqrt{2ab+b^2}$. [*B.U.*]

Suppose, $\sqrt{(a+b+\sqrt{2ab+b^2})}=\sqrt{x}+\sqrt{y}$;

Squaring we have, $a+b+\sqrt{2ab+b^2}=x+y+2\sqrt{xy}$

$\therefore\ x+y=a+b\ldots(1)$ and $2\sqrt{xy}=\sqrt{2ab+b^2}$

$\therefore\ (x+y)^2=(a+b)^2$ and $4xy=2ab+b^2$.

Now, $(x-y)^2=(x+y)^2-4xy=(a+b)^2-2ab-b^2=a^2$.

$\therefore\ x-y=\pm a\ldots\ldots(2)$. Solving (1) and (2) we have

$x=a+\frac{b}{2}=\frac{2a+b}{2}$ and $y=\frac{b}{2}$, or, $x=\frac{b}{2}$ and $y=\frac{2a+b}{2}$.

$\therefore$ The required square root $=\pm\left(\sqrt{\frac{2a+b}{2}}+\sqrt{\frac{b}{2}}\right)$

Ex. 7. Find the square root of $\sqrt{32}-\sqrt{24}$.

$\sqrt{32}-\sqrt{24}=\sqrt{8}(\sqrt{4}-\sqrt{3})=\sqrt{8}(2-\sqrt{3})$

$$=\sqrt{8}\times\frac{4-2\sqrt{3}}{2} \text{ [to make the coefficient of } \sqrt{3}=2\text{]}$$

$$=2\sqrt{2}\times\frac{3+1-2\sqrt{3}}{2}=\sqrt{2}(3+1-2\sqrt{3})=\sqrt{2}(\sqrt{3}-1)^2$$

$$\therefore\ \sqrt{\sqrt{32}-\sqrt{24}}=\pm\sqrt[4]{2}(\sqrt{3}-1).$$

45. Miscellaneous Solutions :

Ex. 1. What must be added to $x^4-6x^3+13x^2-12x+1$ to make it a perfect squar ? [*C. U.* '15]

$$\begin{array}{r|l|l}
 & x^4-6x^3+13x^2-12x+1 & x^2-3x+2 \\
 & x^4 & \\ \hline
2x^2-3x & -6x^3+13x^2-12x+1 & \\
 & -6x^3+\ 9x^2 & \\ \hline
2x^2-6x+2 & 4x^2-12x+1 & \\
 & 4x^2-12x+4 & \\ \hline
 & \qquad\qquad -3 &
\end{array}$$

The expression will be a perfect square, if there is no remainder, *i.e.*, if the last remainder is 0. Here -3 is the last remainder. If we add 3 to it, this remainder becomes 0.

$\therefore$ 3 must be added to the given expression to make it a perfect square.

Ex. 2. What must be subtracted from $4x^4-12x^3-7x^2+25x+14$ to make it a perfect square ?

$$\begin{array}{r|l|l}
 & 4x^4-12x^3-7x^2+25x+14 & 2x^2-3x-4 \\
 & 4x^4 & \\ \hline
4x^2-3x & -12x^3-7x^2+25x+14 & \\
 & -12x^3+9x^2 & \\ \hline
4x^2-6x-4 & -16x^2+25x+14 & \\
 & -16x^2+24x+16 & \\ \hline
 & \qquad\qquad x-2 &
\end{array}$$

Here the last remainder is $x-2$ and the given expression will be a perfect square when the remainder vanishes.

$\therefore$ $x-2$ must be subtracted from the given expression to make it a perfect square.

[N. B. (1) When it is required to find what should be added to make an expression a perfect square, the last remainder with its sign changed is the answer. Thus if the last remainder is $-a+2$ the answer is $a-2$.

(2) When it is required to find what should be subtracted from the expression to make it a perfect square, the last remainder, as it is, is the answer.]

Ex. 3. Find for what value of n will $16x^4-24x^3+41x^2-nx+16$ be a perfect square. [*C. U.* '44]

$$\begin{array}{r|l|l}
 & 16x^4-24x^3+41x^2-nx+16 & 4x^2-3x+4 \\
 & 16x^4 & \\
\hline
8x^2-3x & -24x^3+41x^2-nx+16 & \\
 & -24x^3+\ \ 9x^2 & \\
\hline
8x^2-6x+4 & 32x^2-nx\ +16 & \\
 & 32x^2-24x+16 & \\
\hline
 & 24x-nx &
\end{array}$$

Here, if $24x-nx=0$, the given expression will become a perfect square. $\therefore\ 24x-nx=0,\ \therefore\ nx=24x,\ \therefore\ n=24.$

Ex. 4. Show that $231\times233\times235\times237+16$ is a perfect square.

The given exp. $=(230+1)(230+3)(230+5)(230+7)+16$

$=(x+1)(x+3)(x+5)(x+7)+16$ (putting x for 230)

$=(x^2+8x+7)(x^2+8x+15)+16$

$=(a+7)(a+15)+16$ [putting a for x^2+7x]

$=a^2+22a+121=(a+11)^2$

$=(x^2+8x+11)^2$, which is a perfect square.

Ex. 5. If $(x+1)(x+2)(x+3)(x+4)+c$ is a perfect square, find c.

The given exp. $=(x^2+5x+4)(x^2+5x+6)+c$

$=(a+4)(a+6)+c$ [putting a for x^2+5x] $=a^2+10a+24+c$

$=a^2+2.5.a+(5)^2+c-1=(a+5)^2+c-1.$

Hence, the exp. will be a perfect square, if $c-1=0$. $\therefore\ c=1.$

Ex. 6. Find the condition that x^2+px+q is a perfect square. [*G. U.* '51]

$$\begin{array}{r|l|l}
 & x^2+px+q & x+\frac{p}{2} \\
 & x^2 & \\
\hline
2x+\frac{p}{2} & px+q & \\
 & px+\frac{p}{4}^2 & \\
\hline
 & -\frac{p}{4}+q &
\end{array}$$

Here the given expression will be a perfect square if $-\frac{p^2}{4}+q=0$, *i.e.*, if $p^2=4q$. $\therefore$ The required condition is $p^2=4q$.

Ex. 7. Find the condition that px^2+qx+c may be a perfect square.

$$\begin{array}{r|l|l}
 & px^2+qx+c & x\sqrt{p}+\dfrac{q}{2\sqrt{p}} \\
 & \underline{px^2} & \\
2x\sqrt{p}+\dfrac{q}{2\sqrt{p}} & qx+c & \\
 & \underline{qx+\dfrac{q^2}{4p}} & \\
 & c-\dfrac{q^2}{4p} &
\end{array}$$

$\therefore$ If $c-\dfrac{q^2}{4p}=0$, or, if $c=\dfrac{q^2}{4p}$, or, if $q^2=4pc$, the expression will be a perfect square. $\therefore$ The required condition is $q^2=4pc$.

Exercise 8

Find the square root of the following :—

1. $a^2+2ab+b^2$.
2. $x^4-4x^3y+18x^2y^2-28xy^3+49y^4$. [*C. U.* '33]
3. $x^6+6x^5+x^4-20x^3+28x^2-16x+4$. [*C. U.* '49]
4. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$. [*C. U.* '26]
5. $49x^4+36y^4+109x^2y^2-70x^3y-60xy^3$. [*C. U.* '18]
6. $x^4-6x^3+2ax(x-3)+9x^2+a^2$.
7. $4x^4-12x^3y+25x^2y^2-24xy^3+16y^4$. [*G. U.* '50]
8. $x^4-2ax^3+(a^2+2b^2)x^2-2ab^2x+b^4$. [*C. U.* '48]
9. $a^{-4}+b^{-4}-a^{-2}b^{-2}+2a^{-3}b^{-1}-2a^{-1}b^{-3}$.
10. $4a^4+9a^{-4}+6a^{-2}+4a^2+13$.
11. $x^{\frac{4}{3}}-4x+1+6x^{\frac{2}{3}}-4x^{\frac{1}{3}}$.
12. $4x^{\frac{3}{2}}-12x^{\frac{3}{4}}+25-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}}$.
13. $4x^3+5+x^{-3}-4x^{\frac{3}{2}}-2x^{-\frac{3}{2}}$.
14. $6x^{\frac{4}{3}}+9x^2-4-11x^{\frac{2}{3}}+4x^{-\frac{2}{3}}$.
15. $x^4-x^3-\dfrac{7x^2}{4}+x+1$. [*C. U.* '13]
16. $1+2a+2a^2+a^3+\dfrac{a^4}{4}$. [*C. U.* '17]

17. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. [*C. U.* '51]

18. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$. [*C. U.* '19 ; *Pat.* '18]

19. $x^2-6x+5+\frac{12}{x}+\frac{4}{x^2}$. [*C. U.* '16]

20. $\frac{x^2}{y^2}-\frac{2x}{y}+3-\frac{2y}{x}+\frac{y^2}{x^2}$. [*C· U.* 1909]

21. $x^4-2x^2+3-\frac{2}{x^2}+\frac{1}{x^4}$. [*C. U.* '34]

22. $x^4+\frac{1}{x^2}+2x+6+6x^3+9x^2$. [*C. U.* '50]

23. $x^4+6x^3+11x^2+8x+7+\frac{2}{x}+\frac{1}{x^2}$. [*C. U.* '32]

24. $x^4+4x+2+\frac{4}{x^2}+\frac{4}{x^3}+\frac{1}{x^4}$. [*C. U.* '28, '39, '42]

25. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-\frac{7}{4}$. [*D. B.* '22 ; *B. U.*]

26. $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1$.

27. $\left(a^2+\frac{1}{a^2}\right)-2\left(a+\frac{1}{a}\right)+3$. [*A. U.*]

28. $4a^4+9\left(a^2+\frac{1}{a^2}\right)+12a^2\left(a+\frac{1}{a}\right)+18$. [*Pat.* '28]

29. $x^6+\frac{1}{x^6}-4x^4+4\left(x^2-\frac{1}{x^2}\right)+2$. [*E. B. S. B.* '48 ; *B. U.*]

30. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$. [*C. U.* '46 ; *W. B. S. F.* '52]

31. $(x+2)(x+3)(x+4)(x+5)+1$. [*G. U.* '42]

32. $(x-1)(x-4)(x-7)(x-10)+81$.

33. $(3x^2-11x+6)^2+12x(3x-2)^2$. [*M, U.* 1905]

[Hints :—The exp. $=\{(3x-2)(x-3)\}^2+12x(3x-2)^2$
$=(3x-2)^2(x^2-6x+9)+12x(3x-2)^2$
$=(3x-2)^2(x^2+6x+9)=(3x-2)^2(x+3)^2$ ...]

34. $x^4+y^4+(x+y)^4$. [*Pat.* '26]

[Hints :—The exp. $=(x^2+y^2)^2-2x^2y^2+\{(x+y)^2\}^2=(x^2+y^2)^2$
$-2x^2y^2+(x^2+y^2+2xy)^2=(x^2+y^2)^2-2x^2y^2+(x^2+y^2)^2$
$+4x^2y^2+4xy(x^2+y^2)=2(x^2+y^2)^2+4xy(x^2+y^2)+2x^2y^2$
$=2\{(x^2+y^2)^2+2xy(x^2+y^2)+(xy)^2\}=2(x^2+y^2+xy)^2$
$\therefore$ The square root $=\pm\sqrt{2}(x^2+y^2+xy)$].

35. $\left(\dfrac{x}{x-1}\right)^2-\dfrac{2x^2}{x^2-1}+\left(\dfrac{x}{x+1}\right)^2$.

36. $14-6\sqrt{5}$ 37. $5-2\sqrt{6}$ 38. $4+\sqrt{7}$

39. $33-4\sqrt{35}$ 40. $ax-2a\sqrt{ax-a^2}$

41. $\sqrt{48}-\sqrt{45}$ 42. (i) $\sqrt{252}-\sqrt{245}$,
(ii) $\sqrt{3(4-2\sqrt{3})}$

43. $2x+2\sqrt{x^2-9y^2}$ 44. $12x+2y-4\sqrt{6xy}$

45. Prove that $\sqrt{m+\sqrt{n}}$ cannot be expressed in the form of $\sqrt{x}+\sqrt{y}$ unless m^2-n is a perfect square.

46. Find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square. [*C. U.* '27, '41]

47. For what value of x will $9x^4-12x^3+22x^2-13x+12$ be a perfect square ?

48. Find the numerical value of c which will make $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-2\left(\dfrac{x}{y}+\dfrac{y}{x}\right)+c$ a perfect square. [*C. U.* '50]

49. Find m so that $16x^4-24x^3-mx^2-24x+16$ may be a perfect square. [*Pat.* '26]

50. What must be added to $4a^4-12a^3-7a^2+23a+14$ to make the sum a perfect square ? [*C. U.* '21 ; *W. B. S. F.* '53]

51. What must be added to $x^4-2x^3-3x^2+4x-6$ to make it a perfect square ?

52. What must be taken from $a^4-4a^3+6a^2-5a+2$ to make the remainder a perfect square ?

53. What must be subtracted from $(x+1)(x+2)(x+3)(x+4)+2$ to make it a perfect square ?

54. Find the condition that x^2-px+q may be a perfect square.

55. Find the condition that mx^2+nx+p may be a perfect square.

56. Prove that $33 \times 34 \times 35 \times 36 + 1$ is a perfect square.

57. What least number must be added to $249 \times 247 \times 245 \times 243$ to make it a perfect square? [Ans. 16]

58. Find the first three terms of the square root of $1+a^2$.

59. For what value of a will $x^4+4x^3+8x^2+8x+a^2$ be a perfect square?

60. Find the square root of :—

$$\frac{a^4}{b^4}+\frac{b^4}{a^4}-2\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)+3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-4\left(\frac{a}{b}+\frac{b}{a}\right)+5.$$

Extraction of cube root

46. Here we discuss the process of finding the cube root of an expression. From the formula $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we find that the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$. Now, we have to deduce the method of finding the two terms **a** and **b** of the cube root from the above formula.

First arrange the given expression in the descending powers of a. We find that the first term of it is a^3 of which the cube root is a. So a is put down as the first term of the cube root of the expression. Then subtracting a^3 from the given expression, we have the remainder $3a^2b+3ab^2+b^3$, which may be written in the form $(3a^2+3ab+b^2)\times b$. Then the second term of the cube root is obtained by dividing the remainder by $3a^2+3ab+b^2$.

Now, we observe that here the divisor consists of the following three terms :—

(1) Thrice the square of the first term of the cube root, *i.e.*, $3a^2$;

(2) Thrice the product of the first term and the second term, *i.e.*, $3ab$;

(3) The square of the second term, *i.e.*, b^2.

The whole operation is shown below :—

Ex. 1.

$$\begin{array}{lll|l}
 & & & a^3+3a^2b+3ab^2+b^3\,(a+b \\
 & & & a^3 \\
\hline
3(a)^2=3a^2 & & & 3a^2b+3ab^2+b^3 \\
3\times a\times b= & +3ab & & \\
(b)^2= & & +b^2 & \\
\hline
 & & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
\hline
\end{array}$$

$\therefore$ The required cube root $=a+b$.

Ex. 2. Find the cube root of $8x^3-12x^2+6x-1$.

$$\begin{array}{r|l}
 & 8x^3-12x^2+6x-1\,(2x-1 \\
 & 8x^3 \\ \hline
3(2x)^2=12x^2 \quad & -12x^2+6x-1 \\
3\times 2x\times -1= \quad -6x & \\
(-1)^2= \quad +1 & \\ \hline
12x^2-6x+1 & -12x^2+6x-1 \\ \hline
\end{array}$$

$\therefore$ The required cube root $=2x-1$.

[**N. B.** The second term of the root is found by dividing $-12x^2$ by $12x^2$. We may find it otherwise. The last term -1 of the given expression is a perfect cube. So its cube root -1 must be the second term of the root.]

Ex. 3. Find the cube root of $32-\frac{16}{3a^3}-64a^3+\frac{8}{27a^6}$.

Arrange the expression in the ascending powers of a.

$$\begin{array}{r|l}
 & \frac{8}{27a^6}-\frac{16}{3a^3}+32-64a^3\left(\frac{2}{3a^2}-4a\right. \\
 & \frac{8}{27a^6} \\ \hline
3\left(\frac{2}{3a^2}\right)^2 \quad =\frac{4}{3a^4} & -\frac{16}{3a^3}+32-64a^3 \\
3\times\frac{2}{3a^2}\times(-4a)=-\frac{8}{a} & \\
(-4a)^2 \quad =16a^2 & \\ \hline
\frac{4}{3a^4}-\frac{8}{a}+16a^2 & -\frac{16}{3a^3}+32-64a^3 \\ \hline
\end{array}$$

$\therefore$ The required cube root $=\frac{2}{3a^2}-4a$.

Ex. 4. Find the cube root of 79507.

Here 79507 lies between the two cubes 64000 and 125000, *i.e.*, between $(40)^3$ and $(50)^3$. So the cube root of 79507 must consist of two digits and lie between 40 and 50.

$$\begin{array}{r|l}
 & 79507\,(40+3=43 \\
 & 64000 \\ \hline
3(40)^2=4800 & 15507 \\
3\times 40\times 3= \quad 360 & \\
(3)^2= \quad 9 & \\ \hline
5169 & 15507 \\ \hline
\end{array}$$

$\therefore$ The cube root $=43$.

[**N. B.** Here 7 is the digit in the units' place of the given number, and the cube of no digit other than 3 has 7 in the units' place. $\therefore$ 3 is taken as the second digit of the cube root.]

Ex. 5. Find the cube root of $8a^6+12a^5b-30a^4b^2-35a^3b^3+45a^2b^4+27ab^5-27b^6$.

$$
\begin{array}{rl}
 & 8a^6+12a^5b-30a^4b^2-35a^3b^3+45a^2b^4+27ab^5-27b^6\,\big(2a^2+ab-3b^2 \\
 & 8a^6 \\
\hline
3(2a^2)^2=12a^4 \qquad\qquad & \big|\; 12a^5b-30a^4b^2-35a^3b^3 \\
3\times(2a^2)\times ab= \quad +6a^3b \qquad & \\
(ab)^2= \qquad\qquad +a^2b^2 & \\
\hline
12a^4+6a^3b+a^2b^2 & \big|\; 12a^5b+6a^4b^2+a^3b^3 \\
\hline
3(2a^2+ab)^2=12a^4+3a^2b^2+12a^3b & \big|\; -36a^4b^2-36a^3b^3+45a^2b^4+27ab^5-27b^6 \\
3(2a^2+ab)\times(-3b^2)= \quad -18a^2b^2 \quad -9ab^3 & \\
(-3b^2)^2= \qquad\qquad +9b^4 & \\
\hline
12a^4+12a^3b-15a^2b^2-9ab^3+9b^4 & \big|\; -36a^4b^2-36a^3b^3+45a^2b^4+27ab^5-27b^6 \\
\hline
\end{array}
$$

$\therefore$ The cubic root $=2a^2+ab-3b^2$.

[N. B. Here the first term of the cube root is $2a^2$. Thrice its square is $3(2a^2)^2$ or $12a^4$. The second term in the root is obtained by dividing the first term of the remainder $(12a^5b)$ by $12a^4$. So the second term of the cube root is ab. Then as before we have the first divisor $12a^4+6a^3b+a^2b^2$. Multiplying it by ab and subtracting the product from the first remainder we have the second remainder. Now, thrice the square of $(2a^2+ab)$ of the root is $12a^4+3a^2b^2+12a^3b$. Dividing the first term $(-36a^4b^2)$ of the second remainder by $12a^4$ we get $-3b^2$ as the third term of the cube root. Then the operation is completed as before.]

Ex. 6. For what value of a the expression $a^3+9a^2+28a+25$ may be a perfect cube ?

$$\begin{array}{r|l} & a^3+9a^2+28a+25 \,\big(\, a+3 \\ & a^3 \\ \cline{2-2} 3\times a^2=3a^2 & \qquad 9a^2+28a+25 \\ 3\times a\times 3= \quad 9a & \\ 3\times 3= \quad\ \ 9 & \\ \cline{1-1} 3a^2+9a+9 & \qquad 9a^2+27a+27 \\ \cline{2-2} & \qquad\qquad\quad a-2 \end{array}$$

If the given expression is to be a perfect cube, the remainder $a-2$ must vanish, *i.e.*, $a-2$ must be zero. So $a-2=0$, $\therefore\ a=2$.

Hence the given expression is a perfect cube, when $a=2$.

Exercise 9

Find the cube root of :—

1. $125a^9b^{12}$ 2. $-729x^6y^{-3}$ 3. $-\dfrac{216a^{15}}{343b^{18}}$.

4. $8x^3+12x^2y+6xy^2+y^3$ 5. a^3-3a^2+3a-1

6. $1+3a+6a^2+7a^3+6a^4+3a^5+a^6$

7. $8a^3x^3-27b^3y^3-36a^2x^2by+54axb^2y^2$

8. $54-27x^3+\dfrac{8}{x^6}-\dfrac{36}{x^3}$ 9. $\dfrac{a^3}{8}+\dfrac{3a^2}{4}+\dfrac{3a}{2}+1$

10. 614125 11. $64x^6-125y^6+300x^2y^4-240x^4y^2$

12. $8a^3+b^3+c^3+12a^2b+12abc+6ab^2+12a^2c+6ac^2$
$+3b^2c+3bc^2$.

13. $8a^6+12a^5-30a^4-35a^3+45a^2+27a-27$.

14. $\frac{1}{8}a^3-\frac{1}{4}a^2b+\frac{1}{6}ab^2-\frac{1}{27}b^3$ [*M. U.* 1861]

15. $\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}-3\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^3}\right)+5$ [*B. U.* 1892]

16. $x^3+\dfrac{8}{x^3}-12x^2-\dfrac{48}{x^2}+54x+\dfrac{108}{x}-112$ [*B. U.* 1882]

17. For what value of x will $8x^3-12x^2+7x-3$ be a perfect cube ?

QUADRATIC EQUATIONS

Quadratic equation : The equation in which the highest power of the unknown quantity is 2, is called a **quadratic equation** or an **equation of the second degree** (or, a quadratic). As for example : $x^2=25$, $3x^2+5x=4$, $ax^2+bx+c=0$, etc.

Quadratic equations are of two kinds, pure and adfected.

Pure Quadratic : A quadratic equation having only the second power of the unknown quantity and not the first power is called a **pure quadratic.**

Evidently the coefficient of x in this equation is 0.

Thus $x^2=4$, $ax^2-5=0$ are pure quadratic equations.

Adfected Quadratic : A quadratic equation having both the second and the first powers of the unknown quantity, is called an **adfected quadratic.**

Thus $3x^2-4x=15$ is an adfected quadratic. The general form of such equations is $ax^2+bx+c=0$, where c is called the *constant* term or *absolute* term. Here b and c may have any value other than 0 (zero). If $a=0$, ax^2 vanishes and the equation reduces to a simple equation.

49. Solution of Pure Quadratics

To solve such equations, we have first to transpose the unknown quantities to the left side and the known quantities to the right and then the equation is reduced to the form $x^2=c$. Now taking the square roots of both sides we get the values of the unknown quantity x. As a quantity has two square roots, equal in magnitude but opposite in sign, every quadratic must have two roots. Here $x=\pm\sqrt{c}$. The values of x are called the *roots* of the equation.

Examples 12

Ex. 1. Solve $3x^2-36=64-x^2$.

$3x^2-36=64-x^2$,

or, $3x^2+x^2=64+36$ [by transposition] or, $4x^2=100$, or, $x^2=25$,

$\therefore$ $x=\pm 5$ (*i.e.*, $x=5$ or $x=-5$).

[N. B. Here $x^2=25$, and taking the sq. roots of both sides we get $\pm x=\pm 5$, which is equivalent to $x=\pm 5$.]

Ex. 2. Solve $2x-\frac{3}{x}=\frac{x}{2}$.

Multiplying both sides by $2x$, we have

$$4x^2-6=x^2, \text{ or, } 3x^2=6, \text{ or, } x^2=2, \therefore x=\pm\sqrt{2}.$$

[Otherwise] $\frac{2x^2-3}{x}=\frac{x}{2}$, or, $4x^2-6=x^2$ [by cross-multiplication]

[Now proceed as before]

Ex. 3. Solve $\frac{1}{x+1}+\frac{2}{x+5}=\frac{1}{2}$. [*C. U.* '19]

$$\frac{1}{x+1}+\frac{2}{x+5}=\frac{1}{2}, \text{ or, } \frac{x+5+2x+2}{(x+1)(x+5)}=\frac{1}{2},$$

$$\text{or, } \frac{3x+7}{x^2+6x+5}=\frac{1}{2}, \text{ or, } x^2+6x+5=6x+14,$$

$$\text{or, } x^2=14-5=9, \therefore x=\pm 3.$$

Ex. 4. Solve $\frac{3x+4}{x+2}=\frac{x+5}{x+1}$.

By cross-multiplication we get $3x^2+7x+4=x^2+7x+10$,

or, $2x^2=6$, or, $x^2=3$, $\therefore$ $x=\pm\sqrt{3}$.

Ex. 5. Solve $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$. [*C. U.* '12 ; *D. B.* '22]

From the equn. $\frac{(x+4)^2+(x-4)^2}{(x-4)(x+4)}=\frac{10}{3}$, or, $\frac{2x^2+32}{x^2-16}=\frac{10}{3}$,

or, $10x^2-160=6x^2+96$, or, $4x^2=256$, or, $x^2=64$,

$\therefore$ $x=\pm 8$.

50. Solution of Adfected Quadratics

(1) By Factorization

Ex. 6. Solve $4x^2+25x-351=0$. [*D. B.* '27]

$4x^2+25x-351=0$,

or, $4x^2+52x-27x-351=0$,

or, $4x(x+13)-27(x+13)=0$, or, $(x+13)(4x-27)=0$.

Here, $\because$ the product of two factors is zero, $\therefore$ one of them must be zero. If $x+13=0$, then $x=-13$; and if $4x-27=0$, then $4x=27$, $\therefore$ $x=\frac{27}{4}=6\frac{3}{4}$. $\therefore$ $x=-13$ or $6\frac{3}{4}$.

Ex. 7. Solve $(x-7)(x-19)=64$. [*C. U.* '18]

Simplifying we get $x^2-26x+133=64$, or, $x^2-26x+69=0$,

or, $x^2-23x-3x+69=0$, or, $x(x-23)-3(x-23)=0$,

or, $(x-23)(x-3)=0$.

$\therefore$ either $x-23=0$ or $x-3=0$, $\therefore$ $x=23$ or 3.

Ex. 8. Solve $\frac{x-2}{x+2}+\frac{6(x-2)}{(x-6)}=1$. [*C. U.* '51]

Here, $\frac{6(x-2)}{x-6}=1-\frac{x-2}{x+2}$, or, $\frac{6(x-2)}{x-6}=\frac{x+2-x+2}{x+2}$,

or, $\frac{6(x-2)}{x-6}=\frac{4}{x+2}$, or, $\frac{3(x-2)}{x-6}=\frac{2}{x+2}$,

or, $3(x^2-4)=2(x-6)$, or, $3x^2-12=2x-12$,

or, $3x^2-2x-12+12=0$, or, $3x^2-2x=0$, or, $x(3x-2)=0$,

$\therefore$ either $x=0$ or, $3x-2=0$, $\therefore$ $x=0$ or $\frac{2}{3}$.

Ex, 7. Solve $x+\frac{1}{x}=25\frac{1}{25}$. [*C. U.* '14, '39 Sup.; *D. B.* '25]

$x+\frac{1}{x}=25\frac{1}{25}$, or, $\frac{x^2+1}{x}=\frac{626}{25}$, or, $25x^2+25=626x$,

or, $25x^2-626x+25=0$, or, $25x^2-625x-x+25=0$,

or, $25x(x-25)-1(x-25)=0$, or, $(x-25)(25x-1)=0$,

$\therefore$ either $x-25=0$, or, $25x-1=0$, $\therefore$ $x=25$ or $\frac{1}{25}$.

*Ex. 10. Solve $(x-3)(x-4)=\frac{34}{33^2}$.

Let $a=33$; Now, $(x-3)(x-4)=\frac{34}{a^2}$.

or, $a^2(x-3)(x-4)=34$, or, $a^2(x-3)(x-4)=33+1$.

or, $a(x-3).a(x-4)=a+1$

or, $(ax-3a)(ax-4a)-a-1=0$

or, $(ax-3a)(ax-4a)+(ax-4a)-(ax-3a)-1=0$

$[\because (ax-4a)-(ax-3a)=-a]$

or, $(ax-4a)(ax-3a+1)-1(ax-3a+1)=0$,

or, $(ax-3a+1)(ax-4a-1)=0$,

$\therefore$ either $ax-3a+1=0$, whence $x=\frac{3a-1}{a}=3-\frac{1}{a}=3-\frac{1}{33}=2\frac{32}{33}$;

or, $ax-4a-1=0$, whence $x=\frac{4a+1}{a}=4+\frac{1}{a}=4+\frac{1}{33}=4\frac{1}{33}$.

$\therefore \quad x=2\frac{32}{33}$, or, $4\frac{1}{33}$.

Ex. 11. Solve $\frac{1}{x}-\frac{1}{x+b}=\frac{1}{a}-\frac{1}{a+b}$. [C. U. '21]

Here, $\frac{x+b-x}{x(x+b)}=\frac{a+b-a}{a(a+b)}$, or, $\frac{b}{x^2+bx}=\frac{b}{a^2+ab}$,

or, $x^2+bx=a^2+ab$, or, $x^2+bx-a^2-ab=0$,

or, $(x+a)(x-a)+b(x-a)=0$, or, $(x-a)(x+a+b)=0$,

$\therefore \quad x=a$ or $-(a+b)$.

Ex. 12. Solve $x=\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-x}}}$. [C. U. '33]

Here, $x=\cfrac{1}{2-\cfrac{1}{\cfrac{4-2x-1}{2-x}}}$, or, $x=\cfrac{1}{2-\cfrac{1}{\cfrac{3-2x}{2-x}}}$, or, $x=\cfrac{1}{2-\cfrac{2-x}{3-2x}}$,

$$\text{or, } x=\frac{1}{\dfrac{6-4x-2+x}{3-2x}},\ \text{or, } x=\frac{1}{\dfrac{4-3x}{3-2x}},\ \text{or, } x=\frac{3-2x}{4-3x},$$

or, $4x-3x^2=3-2x$, or, $-3x^2+6x-3=0$,

or, $x^2-2x+1=0$ [dividing by -3]

or, $(x-1)^2=0$, $\therefore$ $x=1, 1$.

[N. B. $(x-1)^2=0$, *i.e.*, $(x-1)(x-1)=0$, $\therefore$ $x=1, 1$.]

Ex. 13. Solve $\dfrac{(x+1)^3-(x-1)^3}{(x+1)^2-(x-1)^2}=2$. [*D. B.* '49]

$$\frac{\{(x+1)-(x-1)\}\{(x+1)^2+(x+1)(x-1)+(x-1)^2\}}{\{(x+1)+(x-1)\}\{(x+1)-(x-1)\}}=2$$

[from the formulas a^3-b^3 and a^2-b^2]

$$\text{or, } \frac{(x+1)^2+(x+1)(x-1)+(x-1)^2}{(x+1)+(x-1)}=2,$$

$$\text{or, } \frac{x^2+2x+1+x^2-1+x^2-2x+1}{2x}=2,\ \text{or, } \frac{3x^2+1}{2x}=2,$$

or, $3x^2+1=4x$, or, $3x^2-4x+1=0$, or, $3x^2-3x-x+1=0$,

or, $(x-1)(3x-1)=0$, $\therefore$ $x=1$ or $\frac{1}{3}$.

✓**Ex. 14.** Solve $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$. [*D. B.* '40, '43, '48]

Here $\dfrac{1}{a+b+x}=\dfrac{bx+ax+ab}{abx}$, or, $(a+b+x)(bx+ax+ab)=abx$,

or, $(a+b+x)(bx+ax+ab)-abx=0$,

or, $(a+b)(a+x)(b+x)=0$, or, $(a+x)(b+x)=0$,

$\therefore$ either $a+x=0$, or $b+x=0$,

$\therefore$ $x=-a$ or $-b$.

(2) Solution by completing the square

Reduce the equation to the general form and then transpose the constant term to the right hand side. Divide both sides by the coefficient of x^2 and then add the square of half the coefficient of x to both sides. Thus the expression on the lefthand side will become a perfect square.

Ex. 15. Solve $x^2-26x=407$. [*D. B.* '29]

$x^2-26x=407$, or, $x^2-26x+(13)^2=407+(13)^2$,

or, $(x-13)^2=407+169=576$, or, $(x-13)=\pm\sqrt{576}=\pm 24$

$\therefore\ x=13\pm 24=37$ or -11.

[**N. B.** Here the coefficient of x is 26 and its half is 13. $\therefore$ $(13)^2$ is added to both sides. Here one root of x is $(13+24)$ or 37 and the other is $(13-24)$ or -11.]

Ex. 16. Solve $10x^2-69x+45=0$. [*D. B.* '30]

$10x^2-69x+45=0$, or, $x^2-\frac{69}{10}x+\frac{45}{10}=0$, or, $x^2-\frac{69}{10}x=-\frac{9}{2}$,

or, $x^2-\frac{69}{10}x+(\frac{69}{20})^2=(\frac{69}{20})^2-\frac{9}{2}$, or, $(x-\frac{69}{20})^2=\frac{4761}{400}-\frac{9}{2}=\frac{2961}{400}$

$$\text{or,}\quad x-\frac{69}{20}=\pm\sqrt{\frac{2961}{400}},\quad \therefore\ x=\frac{69}{20}\pm\frac{\sqrt{2961}}{20}=\frac{69\pm\sqrt{2961}}{20}.$$

Ex. 17. Solve $ax^2+bx+c=0$. [*C. U.* '46]

Dividing both sides by a, we have $x^2+\frac{b}{a}x+\frac{c}{a}=0$,

$$\text{or,}\quad x^2+\frac{b}{a}x=-\frac{c}{a},$$

$$\text{or,}\quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}\quad \text{[adding } \left(\frac{b}{2a}\right)^2 \text{ to both sides]}$$

$$\text{or,}\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},\quad \text{or,}\quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore\quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

[**N. B.** All quadratics can be reduced to the general form $ax^2+bx+c=0$ and then it can be solved from the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Note carefully that in the equation, the coefficient of x^2 is a and that of x is b and the term independent of x is c.

Now in the above formula $-b$ means the coefficient of x with its opposite sign, b^2-4ac denotes the square of the coefficient of x minus 4 times the product of the coefficient of x^2 and the constant term. The denominator is twice the coefficient of x^2.

Here the two roots are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$]

17. (*a*) Apply the above formula to find the roots of the equation $x^2-2\sqrt{3}x-13=0$. [*C. U.* '46]

(3) Solution with the help of the Formula

Ex. **18.** Solve $x^2-\sqrt{17}x-8=0$. [*C. U.* '47]

Here, $x=\dfrac{2\sqrt{17}\pm\sqrt{2\sqrt{17})^2-4\times1\times-8}}{2\times1}$

$$=\frac{2\sqrt{17}\pm\sqrt{68+32}}{2}=\frac{2\sqrt{17}\pm\sqrt{100}}{2}=\frac{2\sqrt{17}\pm10}{2}=\sqrt{17}\pm5.$$

Ex. **19.** Solve $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-3)(x-4)}=\dfrac{1}{6}.$

[*C. U.* '37]

From the equn. $\dfrac{1}{x-2}-\dfrac{1}{x-1}+\dfrac{1}{x-3}-\dfrac{1}{x-2}+\dfrac{1}{x-4}-\dfrac{1}{x-3}=\dfrac{1}{6}$,

or, $\dfrac{1}{x-4}-\dfrac{1}{x-1}=\dfrac{1}{6}$, or $\dfrac{x-1-x+4}{(x-4)(x-1)}=\dfrac{1}{6}$,

or, $\dfrac{3}{x^2-5x+4}=\dfrac{1}{6}$, or, $x^2-5x+4=18$, or, $x^2-5x-14=0$,

or, $(x-7)(x+2)=0$, $\therefore$ $x=7$ or -2.

[N. B. : $\dfrac{1}{(x-1)(x-2)}$ can be written as $\dfrac{1}{x-2}-\dfrac{1}{x-1}$ as $\dfrac{1}{x-2}-\dfrac{1}{x-1}=\dfrac{1}{(x-1)(x-2)}$. The other two given terms are similarly split up.]

Ex. **20.** Find the roots of the quadratic equation $ax^2+2bx+c=0$. [*C. U.* '45 ; *G. U.* '49]

$$ax^2+2bx+c=0, \quad \text{or,} \quad ax^2+2bx=-c,$$

$$\text{or,} \quad x^2+\frac{2b}{a}x=-\frac{c}{a}, \quad \text{or,} \quad x^2+\frac{2b}{a}x+\left(\frac{b}{a}\right)^2=\frac{b^2}{a^2}-\frac{c}{a},$$

$$\text{or,} \quad \left(x+\frac{b}{a}\right)^2=\frac{b^2-ac}{a^2}, \quad \text{or,} \quad x+\frac{b}{a}=\pm\frac{\sqrt{b^2-ac}}{a},$$

$$\therefore \quad x=-\frac{b}{a}\pm\frac{\sqrt{b^2-ac}}{a}=\frac{-b\pm\sqrt{b^2-ac}}{a}.$$

Otherwise :

[From the formula] $x=\dfrac{-2b\pm\sqrt{(2b)^2-4ac}}{2a}$

$$=\frac{-2b\pm\sqrt{4b^2-4ac}}{2a}=\frac{-2b\pm2\sqrt{b^2-ac}}{2a}=\frac{-b\pm\sqrt{b^2-ac}}{a}$$

Ex. **21.** Solve $\dfrac{1}{x+a}+\dfrac{1}{x+2a}+\dfrac{1}{x+3a}=\dfrac{3}{x}$ [*C. U.* '50]

From the given equation,

$$\frac{1}{x+a}-\frac{1}{x}+\frac{1}{x+2a}-\frac{1}{x}+\frac{1}{x+3a}-\frac{1}{x}=0,$$

$$\text{or,} \quad \frac{x-x-a}{x(x+a)}+\frac{x-x-2a}{x(x+2a)}+\frac{x-x-3a}{x(x+3a)}=0,$$

$$\text{or,} \quad \frac{-a}{x(x+a)}+\frac{-2a}{x(x+2a)}+\frac{-3a}{x(x+3a)}=0,$$

$$\text{or,} \quad \frac{1}{x+a}+\frac{2}{x+2a}+\frac{3}{x+3a}=0, \quad \text{or,} \quad \frac{1}{x+a}+\frac{3}{x+3a}=\frac{-2}{x+2a},$$

$$\text{or,} \quad \frac{x+3a+3x+3a}{(x+a)(x+3a)}=\frac{-2}{x+2a}, \quad \text{or,} \quad \frac{4x+6a}{x^2+4ax+3a^2}=\frac{-2}{x+2a},$$

$$\text{or,} \quad \frac{2x+3a}{x^2+4ax+3a^2}=\frac{-1}{x+2a},$$

$$\text{or,} \quad 2x^2+7ax+6a^2=-x^2-4ax-3a^2,$$

$$\text{or,} \quad 3x^2+11ax+9a^2=0,$$

$$\therefore \quad x=\frac{-11a\pm\sqrt{(11a)^2-4.3.9a^2}}{6}=\frac{-11a\pm\sqrt{13a^2}}{6}$$

$$=\frac{-11a\pm a\sqrt{13}}{6}=\frac{-11\pm\sqrt{13}}{6}\cdot a.$$

Ex. 22. Solve $\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}=0.$

[*C. U.* '26, '29 ; *D. B.* '50]

Multiplying by $(x-a)(x-b)(x-c)$ we get

$$(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0,$$

or, $x^2-(b+c)x+bc+x^2-(c+a)x+ac+x^2-(a+b)x+ab=0,$

or, $3x^2-(b+c+c+a+a+b)x+ab+bc+ca=0,$

or, $3x^2-2(a+b+c)x+(ab+bc+ca)=0,$

$$\therefore \quad x=\frac{2(a+b+c)\pm\sqrt{4(a+b+c)^2-4.3.(ab+bc+ca)}}{6}$$

$$=\frac{2(a+b+c)\pm2\sqrt{(a+b+c)^2-3(ab+bc+ca)}}{6}$$

$$=\frac{(a+b+c)\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

Ex. 23. Solve, without assuming any formula, the equation $x^2-11x=82052.$

[*C. U.* '42]

$x^2-11x=82052$, or, $x^2-11x+(\frac{11}{2})^2=82052+(\frac{11}{2})^2$,

or, $(x-\frac{11}{2})^2=82052+\frac{121}{4}=\frac{328329}{4}$,

or, $x-\frac{11}{2}=\pm\frac{573}{2}$

$\therefore \quad x=\frac{11}{2}\pm\frac{573}{2}=292$ or $-281.$

(4) The Hindu Method or Sridhar Acharya's method

Reduce the equation to the general form $ax^2+bx+c=0$, and transpose the constant term c to the right side. Now, multiply both sides by $4a$ (*i.e.*, 4 times the coefficient of x^2) and add b^2 (*i.e.*, the square of the coefficient of x) to both sides. The lefthand side will then become a perfect square.]

Ex. 24. Solve $3x^2-11x+9=0$. [*C. U.* '35]

Here $3x^2-11x=-9$,

or, $36x^2-132x=-108$ [Multiplying by 4 × 3]

or, $36x^2-132x+(11)^2=121-108$ [Adding $(-11)^2$ to both sides]

or, $(6x-11)^2=13$, or, $6x-11=\pm\sqrt{13}$,

or, $6x=11\pm\sqrt{13}$, $\therefore\ x=\dfrac{11\pm\sqrt{13}}{6}$.

Ex. 25. Solve $x^2-2\sqrt{7}x-2=0$. [*G. U.* '48]

Here $x^2-2\sqrt{7}x=2$,

or, $4x^2-8\sqrt{7}x=8$ [Multiplying by 4 × 1]

or, $4x^2-8\sqrt{7}x+(2\sqrt{7})^2=(2\sqrt{7})^2+8$

[Adding $(2\sqrt{7})^2$ to both sides]

or, $(2x-2\sqrt{7})^2=28+8=36$,

or. $2x-2\sqrt{7}=\pm 6$, or, $2x=2\sqrt{7}\pm 6$, $\therefore\ x=\sqrt{7}\pm 3$.

51. Exponential equations

Ex. 1. Solve $4\times 2^{x-1}=8^x$.]*D. B.* '31]

$4\times 2^{x-1}=8^x$, or, $2^2\times 2^{x-1}=(2^3)^x$,

or, $2^{x-1+2}=2^{3x}$, or, $2^{x+1}=2^{3x}$,

$\therefore\ 3x=x+1$, or, $2x=1$, $\therefore\ x=\frac{1}{2}$.

[**N. B.** Here 4 and 8 are expressed in powers of 2 as $4=2^2$ and $8=2^3$. Now, since 2^{x+1} and 2^{3x} are equal, their indices $x+1$ and $3x$ must be equal, the bases being the same. $\therefore\ 3x=x+1$.]

Ex. 2. Solve $2^{x+1}=\sqrt[x]{64}$.

$2^{x+1}=\sqrt[x]{64}$, or, $2^{x+1}=65^{\left(\frac{1}{x}\right)}$

or, $2^{x+1}=(2^6)^{\frac{1}{x}}$ [Expressing 64 in terms of the base 2] or, $2^{x+1}=2^{\frac{6}{x}}$, $\therefore\ x+1=\frac{6}{x}$, or, $x^2+x=6$,

or, $x^2+x-6=0$, or, $(x+3)(x-2)=0$.

$\therefore\ x=2$, or, -3.

Ex. 3. Solve $(\sqrt{3})^{2x+4}=243$. [*C. U.* '32, '50 ; *G. U.* '51]

$(\sqrt{3})^{2x+4}=243$, or, $\left(3^{\frac{1}{2}}\right)^{2x+4}=(3)^5$ $[\because\ 243=3^5]$
or, $3^{x+2}=3^5$. $\therefore\ x+2=5$, $\therefore\ x=3$.

Ex. 4. Solve $9^x=\frac{9}{3^x}$. [*P. U.* '30]

$9^x=\frac{9}{3^x}$, or, $(3^2)^x=\frac{9}{3^x}$, or, $3^{2x}=\frac{9}{3^x}$, or, $3^{2x}.3^x=9$,

or, $3^{3x}=3^2$, $\therefore\ 3x=2$, $\therefore\ x=\frac{2}{3}$.

Ex. 5. Solve $4^{x^2+x+2}\times 5^{x^2+x+3}=800000$.

$$4^{x^2+x+2}\times 5^{x^2+x+3}=800000,$$

or, $4^{x^2+x+2}\times 5^{x^2+x+2}\times 5=800000$,

or, $4^{x^2+x+2}\times 5^{x^2+x+2}=160000$ [Dividing both sides by 5]

or, $(20)^{x^2+x+2}=(20)^4$, $\therefore\ x^2+x+2=4$,

or, $x^2+x-2=0$, or, $(x-1)(x+2)=0$, $\therefore\ x=1, -2$.

Ex. 6. Solve $4.3^{x+1}=27+9^x$. [*C. U.* '51]

$$4.3^{x+1}=27+9^x, \text{ or, } 4.3^x.3=27+3^{2x},$$

or, $12.3^x=27+(3^x)^2$, or, $12p=27+p^2$ [putting p for 3^x]

or, $p^2-12p+27=0$, or, $(p-9)(p-3)=0$, $\therefore\ p=9$ or 3.

If $p=9$, we have $3^x=9=3^2$, $\therefore\ x=2$; and

if $p=3$, we have $3^x=3=3^1$, $\therefore\ x=1$. $\therefore\ x=2$ or 1.

Ez. 7. Solve $2^{x+4}+2^{x+1}=144$.

$$2^{x+4}+2^{x+1}=144, \text{ or, } 2^x.2^4+2^x.2=144,$$

or, $16.2^x+2.2^x=144$, or, $18.2^x=144$, or, $2^x=8=2^3$, $\therefore\ x=3$.

Ex. 8. Solve $a^{2x-3}=b^{2x-3}$.

$a^{2x-3}=b^{2x-3}$, $\therefore$ $\dfrac{a^{2x-3}}{b^{2x-3}}=1$, or, $\left(\dfrac{a}{b}\right)^{2x-3}=1=\left(\dfrac{a}{b}\right)^{0}\left[\because\left(\dfrac{a}{b}\right)^{0}=1\right]$

$\therefore$ $2x-3=0$, $\therefore$ $x=\frac{3}{2}$.

Ex. 9. Solve $\left.\begin{aligned} 3^x.9^y &= 27^z\cdots(1)\\ 4^x.8^y &= 32^z\cdots(2)\\ 2^x.5^y,7^z &= 70\cdots(3)\end{aligned}\right\}$ [*C. U.* '43]

From (1) we get $3^x.3^{2y}=3^{3z}$, or, $3^{x+2y}=3^{3z}$,

$\therefore$ $x+2y=3z$, or, $x+2y-3z=0\cdots(4)$

From (2) ,, ,, $2^{2x}.2^{3y}=2^{5z}$, or, $2^{2x+3y}=2^{5z}$,

$\therefore$ $2x+3y=5z$, or, $2x+3y-5z=0\ldots(5)$

From (4) and (5) by method of cross-multiplication we get

$$\frac{x}{-10+9}=\frac{y}{-6+5}=\frac{z}{3-4},\ \text{or,}\ \frac{x}{-1}=\frac{y}{-1}=\frac{z}{-1},$$

or, $x=y=z=k$ (suppose)

Now from (3) we get $2^k.5^k.7^k=70$, or, $(2.5.7)^k=70$,

or, $70^k=70^1$, $\therefore$ $k=1$.

$\therefore$ $x=y=z=1$.

Ex. 10. Solve $x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0$. [*C. U.* '30]

$$x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0,\ \text{or,}\ (x^{\frac{1}{3}})^2+x^{\frac{1}{3}}-2=0,$$

or, $a^2+a-2=0$ [putting a for $x^{\frac{1}{3}}$]

or, $(a-1)(a+2)=0$, $\therefore$ $a=1$ or -2, $\therefore$ $x^{\frac{1}{3}}=1$ or -2.

$\therefore$ $(x^{\frac{1}{3}})^3=(1)^3$ or $(-2)^3$, $\therefore$ $x=1$ or -8.

✓Ex. 11. Solve $5^x+5^{-x}=25\frac{1}{25}$.

Now, $5^x+\dfrac{1}{5^x}=25+\dfrac{1}{25}$, or, $5^x+\dfrac{1}{5^x}=5^2+\dfrac{1}{5^2}$,

or, $5^x-5^2-\dfrac{1}{5^2}+\dfrac{1}{5^x}=0$, or, $(5^x-5^2)-\left(\frac{1}{5^2}-\frac{1}{5^x}\right)=0$, or, $(5^x-5^2)-\left(\frac{5^x-5^2}{5^2.5^x}\right)=0$,

or, $(5^x-5^2)\left(1-\frac{1}{5^2.5^x}\right)=0$,

∴ $5^x-5^2=0$...(1), or, $1-\frac{1}{5^2.5^x}=0$...(2)

Now, from (1) we have $5^x=5^2$, $x=2$;

and form (2) ,, ,, $\frac{1}{5^x.5^2}=1$, or, $\frac{1}{5^x}=5^2=\frac{1}{5^{-2}}$,

∴ $x=-2$. Hence $x=2$ or -2.

Ex. 12. Solve $a^{2x}.a^{y+1}=a^8$, and $a^{3y}.a^{3x+5}=a^{20}$.

∵ $a^{2x}.a^{y+1}=a^8$ ∴ $a^{2x+y+1}=a^8$,

∴ $2x+y+1=8$, or, $2x+y=7$...(1)

Again, ∵ $a^{3y}.a^{3x+5}=a^{20}$, ∴ $a^{3x+3y+5}=a^{20}$,

∴ $3x+3y+5=20$, or, $3x+3y=15$···(2)

Solving (1) and (2) we get $x=2$, $y=3$.

52. Irrational Equations

Ex. 1. Solve $\sqrt{x+6}+\sqrt{x-2}=4$. [*C. U.* '47]

$\sqrt{x+6}+\sqrt{x-2}=4$, or, $\sqrt{x+6}=4-\sqrt{x-2}$,

or, $x+6=16+x-2-8\sqrt{x-2}$ [Squaring both sides]

or, $8\sqrt{x-2}=16-2-6=8$, or, $\sqrt{x-2}=1$, or, $x-2=1$

∴ $x=3$.

Ex. 2. Solve $\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=5$.

By componendo & dividendo we have,

$\frac{2\sqrt{a^2+x^2}}{2\sqrt{a^2-x^2}}=\frac{5+1}{5-1}$, or, $\frac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}=\frac{6}{4}=\frac{3}{2}$,

or, $\frac{a^2+x^2}{a^2-x^2}=\frac{9}{4}$ [Squaring], or, $\frac{2a^2}{2x^2}=\frac{13}{5}$ [by comp. & div.]

or, $\frac{a^2}{x^2}=\frac{13}{5}$, or, $13x^2=5a^2$, or, $x^2=\frac{5}{13}a^2$, ∴ $x=\pm a\sqrt{\frac{5}{13}}$.

Ex. 3. Solve $2(x+2)=1+\sqrt{4x^2+9x+14}$. [*C. U.*]

We have $2x+4=1+\sqrt{4x^2+9x+14}$,

or, $2x+3=\sqrt{4x^2+9x+14}$,

or, $4x^2+12x+9=4x^2+9x+14$ [Squaring]

or, $3x=5$, $\therefore$ $x=\frac{5}{3}$.

Ex. 4. Solve $2-\sqrt{11x^2+5x+3}=x$.

Here, $-\sqrt{11x^2+5x+3}=x-2$,

or, $11x^2+5x+3=x^2-4x+4$ [Squaring]

or, $10x^2+9x-1=0$, $\therefore$ $10x^2+10x-x-1=0$,

or, $(10x-1)(x+1)=0$, $x=\frac{1}{10}$ or -1.

[See the notes under Example 10]

Ex. 5. Solve $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$.

By cross-multiplication we have $x^2-ax^2=x$,

or, $x^2(1-a)-x=0$, or, $x\{x(1-a)-1\}=0$,

$\therefore$ Either $x=0$, or $x(1-a)-1=0$, $\therefore$ $x=0$, or $\dfrac{1}{1-a}$.

✓Ex. 6 Solve $\dfrac{ax-1}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}$,

$\because$ $ax-1=(\sqrt{ax})^2-(1)^2=(\sqrt{ax}+1)(\sqrt{ax}-1)$,

$\therefore$ From the given equation, $\dfrac{(\sqrt{ax}+1)(\sqrt{ax}-1)}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}$,

or, $\sqrt{ax}-1=4+\dfrac{\sqrt{ax}-1}{2}$, or, $(\sqrt{ax}-1)-\dfrac{\sqrt{ax}-1}{2}=4$,

or, $\dfrac{\sqrt{ax}-1}{2}=4$, or, $\sqrt{ax}-1=8$, or, $\sqrt{ax}=9$,

$\therefore$ $ax=81$, $\therefore$ $x=\dfrac{81}{a}$.

Ex. 7. Solve $\sqrt{x}+\sqrt{x+4}=\dfrac{2}{\sqrt{x}}$. [*C. U.*]

$x+\sqrt{x^2+4x}=2$ (by cross-multiplication)

or, $\sqrt{x^2+4x}=2-x$, or, $x^2+4x=4-4x+x^2$,

or, $8x=4$, $\therefore$ $x=\frac{1}{2}$.

Ex. 8. Solve $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}.$ [*C. U.* '27]

Squaring both sides, $x^2+9+x^2-9+2\sqrt{x^4-81}$
$$=16+34+8\sqrt{34},$$

or, $2x^2+2\sqrt{2x^4-81}=50+8\sqrt{34}.$

Now, equating the rational parts of the two sides we have $2x^2=50$, or, $x^2=25$, $\therefore$ $x=\pm5.$

Ex. 9. Solve $\sqrt{3x^2+16}-\sqrt{3x^2-16}=8-4\sqrt{2}\cdots(1)$

[**Alt. method**] $(3x^2+16)-(3x^2-16)=32$ [Identically],

i.e., $(\sqrt{3x^2+16})^2-(\sqrt{3x^2-16})^2=32\ldots\ldots\ldots\ldots(2)$

Dividing (2) by (1) we get $\sqrt{3x^2+16}+\sqrt{3x^2-16}=\dfrac{32}{8-4\sqrt{2}}$

or, $\sqrt{3x^2+16}+\sqrt{3x^2-16}=\dfrac{32(8+4\sqrt{2})}{(8)^2-(4\sqrt{2})^2}=\dfrac{32(8+4\sqrt{2})}{32}$
$$=8+4\sqrt{2}\cdots(3)$$

Now, from (1)+(3) we get $2\sqrt{3x^2+16}=16$, or, $\sqrt{3x^2+16}=8$,

$\therefore$ $3x^2+16=64$, or, $x^2=16$, $\therefore$ $x=\pm4.$

Ex. 10. Solve $4x^2+6x+\sqrt{2x^2+3x+4}=13.$ [*W.B.S.F.* '53]

or, $4x^2+6x+8+\sqrt{2x^2+3x+4}=21$ [Adding 8 to both sides]

or, $2(2x^2+3x+4)+\sqrt{2x^2+3x+4}=21,$

or, $2a+\sqrt{a}=21$ [putting a for $2x^2+3x+4$]

or, $2a-21=-\sqrt{a}$, or, $4a^2-84a+441=a$, [squaring]

or, $4a^2-85a+441=0$, or, $4a^2-36a-49a+441=0,$

or, $(a-9)(4a-49)=0$, $\therefore$ $a=9$, or $\frac{49}{4}.$

Now, if $a=9$, then $2x^2+3x+4=9$, or, $2x^2+3x-5=0,$

or, $(x-1)(2x+5)=0$, $\therefore$ $x=1$ or $-\frac{5}{2}.$

Again, if $a=\frac{49}{4}$, then $2x^2+3x+4=\frac{49}{4},$

or, $8x^2+12x+16=49$, or, $8x^2+12x-33=0,$

$$\therefore\ x=\frac{-12\pm\sqrt{144+1056}}{16}=\frac{-12\pm20\sqrt{3}}{16}=\frac{-3\pm5\sqrt{3}}{4}$$

Hence $x=1,\ -\dfrac{5}{2}$, or $\dfrac{-3\pm5\sqrt{3}}{4}.$

Here we find that the roots 1 and $-\frac{5}{2}$ satisfy the equation, but the other roots $\frac{-3 \pm 5\sqrt{3}}{4}$ do not. Hence the roots $\frac{-3 \pm 5\sqrt{3}}{4}$ are not admissible.

∴ The required roots of the equation are $x=1$ or $-\frac{5}{2}$.

[N. B. In the above equation the roots $\frac{-3 \pm 5\sqrt{3}}{4}$ are called **extraneous roots.** These roots satisfy the equation $4x^2+6x-\sqrt{2x^2+3x+4}=13$, but not the given equation.

When equations involving surds are solved by squaring (or clearing the radicals) both sides, all the roots obtained may not satisfy the equation. So in every such solution the roots obtained should be carefully tested. Only those roots that satisfy it should be given as the required roots and the other roots should be stated as extraneous roots.

In verifying such cases only the positive square roots are to be considered.]

Ex. 11. Solve $\frac{x^2-3x-24}{x^2-9}+\frac{4}{x-3}=1\frac{1}{5}$.

Multiplying both sides by $5(x^2-9)$ or $5(x+3)(x-3)$ we get

$$5(x^2-3x-24)+20(x+3)=6(x^2-9) \cdots\cdots (1)$$

or, $5x^2-15x-120+20x+60=6x^2-54$,

or, $x^2-5x+6=0$, or, $(x-2)(x-3)=0$, ∴ $x=2$ or 3.

Here we find that the root 2 satisfies the equation, but the root 3 does not. Putting $x=3$ in the equation we have

$\frac{-24}{0}+\frac{4}{0}=1\frac{1}{5}$ which is absurd. ∴ 3 is an extraneous root here.

∴ The required solution is $x=2$.

[*Notes* : In the above solution we have multiplied both sides by $5(x+3)(x-3)$ and taken the products to be equal, but the process will be absurd if $x-3=0$.]

Miscellaneous Solutions

Ex. 1. Solve $x^2+\frac{36}{x^2}=13$. [*C. U.* '31]

From the equation, $x^4+36=13x^2$, or, $x^4-13x^2+36=0$,
or, $(x^2-9)(x^2-4)=0$, or, $(x-3)(x+3)(x-2)(x+2)=0$,
$\therefore\ x=\pm 3$ or ± 2.

Ex. 2. Solve $x^6-28x^3+27=0$. [*D. U.* '31]

From the equation, $x^6-27x^3-x^3+27=0$,
or, $x^3(x^3-27)-1(x^3-27)=0$, or, $(x^3-1)(x^3-27)=0$,
or, $(x-1)(x^2+x+1)(x-3)(x^2+3x+9)=0$,
$\therefore\ x-1=0\ldots(1)$, or, $x^2+x+1=0\ldots(2)$, or, $x-3=0\ldots(3)$
or, $x^2+3x+9=0\ldots(4)$. Now from (1) we have $x=1$;

from (2), we have $x=\dfrac{-1\pm\sqrt{1^2-4}}{2}=\dfrac{-1\pm\sqrt{-3}}{2}$;

from (3), $x=3$; from (4), $x=\dfrac{-3\pm\sqrt{9-36}}{2}$

$$=\frac{-3\pm\sqrt{-27}}{2}=\frac{-3\pm 3\sqrt{-3}}{2}=\tfrac{3}{2}(-1\pm\sqrt{-3}).$$

Hence, $x=1,\ \dfrac{-1\pm\sqrt{-3}}{2},\ 3,\ \tfrac{3}{2}(-1\pm\sqrt{-3})$.

[**N. B.** In the roots of the above equation $\sqrt{-3}$ means the square root of a negative quantity but this operation is not possible. So $\sqrt{-3}$ has no real value. But as such quantities are often used in algebra, they are called **imaginary**. If in quadratic equations b^2-4ac is negative, such roots will occur.]

Ex. 3. Solve $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}$. [*C.U.*, ; *D.B.* '46, '51]

Cubing both sides, we have

$1+x+1-x+3(1+x)^{\frac{1}{3}}(1-x)^{\frac{1}{3}}\{(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}\}=2$,

or, $2+3(1-x^2)^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=2$ [$\because\ \{(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}\}=2^{\frac{1}{3}}$]

or, $3(1-x^2)^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=0$, $\therefore\ (1-x^2)^{\frac{1}{3}}=0$, $\therefore\ 1-x^2=0$,
or, $x^2=1$, $\therefore\ x=\pm 1$.

Exercise 10

Solve :—

1. $5x^2+3=128.$ **2.** $(2x-1)^2=5-4x.$

3. $2x-\frac{3}{x}=\frac{x}{2}.$ **4.** $\frac{2x+1}{x+1}=\frac{x+8}{x+4}.$ [*C. U.* '31]

5. $42x^2-41x-20=0.$ [*C. U.* '13]

6. $3x^2-10x+3=0.$ [*C. U.* '33]

7. $4x^2-65x+126=0.$ [*C. U.* '16]

8. $6x^2-11x-10=0.$ [*C. U.* '22]

9. $(17x-8)(x-2)=555.$ [*C. U.* '32]

10. $17x^2+19x=1848.$ [*C. U.* '11]

11. $6x^2-91x+323=0.$ [*C. U.* '14]

12. $(x-7)(x-19)=64.$ [*C. U.* '18]

13. $(x+4)(2x-3)=6.$ [*B. U.* '29]

14. $x^2-6x+2=0.$ [*G. U.* '48]

15. $x^2-10x+8=0$ [*C. U.* '47]

16. $x^2+2(b-c)x+c^2=2bc.$

17. $ax^2-bx-c=0.$ [*C. U.* '44]

18. $x+\frac{1}{x}=6\frac{1}{6}.$ 19. $\frac{x}{3}+\frac{3}{x}=4\frac{1}{4}.$ [*C. U.* '31]

20. $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{6}.$ [*E. B. S. B.* '50]

21. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{25}{12}.$ [*C. U.* '10]

22. $\frac{x+3}{x-3}+6\frac{x-3}{x+3}=5.$ [*W.B.S.F.* '52]

23. $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{6}{7}=0.$ [*C. U.* '11]

24. $\frac{12x+17}{3x+1}-\frac{2x+15}{x+7}=3\frac{1}{5}.$ [*C. U.* '20]

25. $\dfrac{40}{x-5}+\dfrac{27}{x}=13.$ [*D. B.* '26]

26. $\dfrac{x-6}{x+2}+\dfrac{x-10}{x+6}+2=0.$ [*C. U.* '28]

27. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{2x+13}{x+1}.$

28. $\dfrac{x+1}{2}+\dfrac{2}{x+1}=\dfrac{x+1}{3}+\dfrac{3}{x+1}-\dfrac{5}{6}.$ [*C. U.* '36]

29. $\dfrac{x-3}{x+3}+\dfrac{x+3}{x-3}=\dfrac{2(x+4)}{x-4}.$

30. $\left(\dfrac{x-a}{x+a}\right)^2-5\left(\dfrac{x-a}{x+a}\right)+6=0.$ [*P. U.*]

31. $1+x=\dfrac{3}{4-\dfrac{3}{4-x}}.$ [*C. U.* '44]

32. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}.$ [*D. B.* '40, '43]

33. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a}{b}+\dfrac{b}{a}.$ [*P. U.* 1891]

34. $x^2-2\sqrt{13}x+4=0$ [*C. U.* '49]

35. $x^2-2\sqrt{3}x-13=0$ [*C. U.* '46]

36. $\sqrt{x+2}+\sqrt{x-3}=5$ [*C. U.* 29, '41]

37. $\sqrt{x+3}+\sqrt{x-2}=5.$ [*C. U.* '46]

38. $\sqrt{5x-1}=1+\sqrt{5x-2}.$ [*C. U.*]

39. $\sqrt{x+1}+\sqrt{x+8}=\sqrt{6x+1}.$ [*C. U.* '20 ; *D. B.* '30]

40. $\sqrt{2x-9}-\sqrt{x-4}=\sqrt{x+1}.$

41. $\sqrt{a+x}+\sqrt{x}=\dfrac{ma}{\sqrt{a+x}}.$

42. $\sqrt{2x^2+9}+\sqrt{2x^2-9}=9+3\sqrt{7}.$

43. $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$ [*C. U.* '50 ; *Pat. U.* '21]

44. $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3.$ [*C. U.*]

45. $\dfrac{x-9}{\sqrt{x}+3}=3+\dfrac{3-\sqrt{x}}{2}.$

46. $\sqrt{\dfrac{x-a}{x-b}}+\dfrac{a}{x}=\sqrt{\dfrac{x-b}{x-a}}+\dfrac{b}{x}.$ [*M. U.*]

47. $x^{\frac{2}{3}}+x^{\frac{1}{3}}=12.$ 48. $x^{-3}+x^{-\frac{3}{2}}=2.$

49. $4\times 2^{x-1}=8^x.$ [*D. B.* '31]

50. $4^x+2^{x+2}=96.$ [*D. B.* '47]

51. $(\sqrt{2})^{3x+1}=256.$ 52. $3.2^{x+1}=8+4^x.$

53. $a^{2x-1}=b^{2x-1}.$ 54. $\left(\dfrac{p}{q}\right)^{x-2}=\left(\dfrac{q}{p}\right)^{2x-1}.$

55. $(\sqrt[3]{36})^{2x+1}=(\sqrt[5]{216})^{x+6}.$ 56. $2^{x-2}+2^{3-x}=3$ [*C. U.* '40]

57. $2(4^x-3.2^{x-1})+1=0.$ 58. $5^{x-2}+5^{1-x}=1\frac{1}{5}.$

59. $2^{2x-2}=2^{x-3}-\frac{1}{64}.$ 60. $12^x+12^{-x}=144\frac{1}{144}.$

61. $4^x-3.2^{x+2}+32=0$ [*W. B. S. F.* '53]

62. $3^{x+y}=3^{2x-y}=\sqrt{27}.$

63. $4^{x+\frac{1}{2}}+4=36(2^{x-2}).$ [*W. B. S. F.* '52]

64. $9(9^{x-1}+3)=28\times 3^x.$

65. $4^{\frac{x+3}{2}}.8^{\frac{x-3}{3}}=2^{\frac{5x+7}{3}}.16^{\frac{3x-1}{7}}.$

66. $$\left.\begin{aligned}\frac{5^x}{5^y}&=25\\ \frac{4^y}{2^x}&=2\end{aligned}\right\}$$ [*G. U.* '54]

67. $5^{x^2+x+1}6^{x^2+x+2}=162000.$

68. $a^x.a^{y+1}=a^7,\ a^{2y}.a^{3x+5}=a^{20}.$ [*C. U.* 1878]

69. $6(x^4+1)=13x^2.$ [*D. B.* '32]

70. $3x^{\frac{1}{2}}-\frac{8}{x^{\frac{3}{2}}}+\frac{10}{x^{\frac{1}{2}}}=0.$ [*D. B.* '35]

✓71. $(p+x)^{\frac{2}{3}}+(p-x)^{\frac{2}{3}}=3(p^2-x^2)^{\frac{1}{3}}.$

Solve without assuming any formula :—

72. $3x^2+4x=8.$ [*C. U.* '51]

73. $x^2-x=1806.$ [*C. U.* '17] 74. $63x^2-62x=221.$

75. Find the value of x for which $ax^2-(a+1)x+1=0.$

76. Solve $15x^2-11x-31=0$ and find the values of the roots correct to 2 decimal places. [*G. U.* '54]

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
CALCUTTA-27

PROBLEMS ON QUADRATIC EQUATIONS

. **53.** In solving problems on quadratic equations, it may so happen that both the obtained roots of the equation do not satisfy the given condition. In such cases the root that is incompatible with the condition of the problem should be rejected.

Examples 13

Ex. 1. Find two consecutive numbers the sum of whose squares is 145. [*C. U.* '16]

Let x and $x+1$ be the two consecutive numbers.

From the given condition we have $x^2+(x+1)^2=145$,

or, $x^2+x^2+2x+1=145$, or, $2x^2+2x-144=0$.

or, $x^2+x-72=0$, or, $(x+9)(x-8)=0$, $\therefore$ $x=-9$ or 8.

$\therefore$ The required numbers are 8 and 9.

This is the arithmetical solution of the problem.

In the Algebraic solution, however, -9 and -8 (obtained from $x=-9$) are admissible.

$\therefore$ the numbers are 8 and 9, or, -9 and -8.

Ex. 2. Find two consecutive odd numbers whose product is 899. [*Pat. U.* '24]

Let $2x-1$ and $2x+1$ be the two consecutive odd numbers.

Then by the question we have $(2x-1)(2x+1)=899$,

or, $4x^2-1=899$, or, $4x^2=900$, or, $x^2=225$, $\therefore$ $x=\pm 15$.

$\therefore$ The required numbers $=29$ and 31, or, -31 and -29.

[N. B. Since $2x$ is an even number, whatever integral value x may have, $2x+1$ or $2x-1$ must be odd numbers.]

Ex. 3. A man bought a certain number of books for Rs. 20. Had he obtained one more book for the same, the average price of each would have been a rupee less. Find the number of books bought.

Let x be the number of books bought, then the average price of each book is $\frac{20}{x}$ rupees.

Had he bought $x+1$ books for Rs. 20, the average price would have been $\frac{20}{x+1}$ rupees. $\therefore$ From the given condition we have

$$\frac{20}{x+1}=\frac{20}{x}-1, \text{ or, } \frac{20}{x+1}=\frac{20-x}{x},$$

or, $20x=20x+20-x^2-x$, or, $x^2+x-20=0$,
or, $(x+5)(x-4)=0$, $\therefore x=-5$ or 4.

The negative value is inadmissible as the number of books cannot be negative. $\therefore$ The required number of books $=4$.

Ex. 4. Divide unity into two parts such that the sum of their cubes is $\frac{7}{16}$. [*C. U.* '15]

Let x be one part. Then the other part is $1-x$.
$\therefore$ by the problem, $(x)^3+(1-x)^3=\frac{7}{16}$,
or, $x^3+1-3x+3x^2-x^3=\frac{7}{16}$, or, $3x^2-3x+\frac{9}{16}=0$,
or, $48x^2-48x+9=0$, or $16x^2-16x+3=0$,
or, $16x^2-12x-4x+3=0$, or, $(4x-1)(4x-3)=0$,
$\therefore x=\frac{1}{4}$ or $\frac{3}{4}$. $\therefore$ The required parts $=\frac{1}{4}$ and $\frac{3}{4}$.

Ex. 5. Divide 50 into two parts such that the sum of their reciprocals may be $\frac{1}{12}$. [*C. U.* '13]

Suppose one part is x. Then the other part is $50-x$.

$\therefore$ from the given condition we have, $\frac{1}{x}+\frac{1}{50-x}=\frac{1}{12}$,

or, $\frac{50-x+x}{x(50-x)}=\frac{1}{12}$, or, $\frac{50}{50x-x^2}=\frac{1}{12}$, or, $600=50x-x^2$,
or, $x^2-50x+600=0$, or, $x^2-30x-20x+600=0$,
or, $(x-30)(x-20)=0$, $\therefore x=30$ or 20.

Hence the required parts are 20 and 30.

[N. B. If the product of two numbers be 1, each is called the reciprocal of the other. Thus $\frac{5}{4}$ is the reciprocal of $\frac{4}{5}$, 5 is the reciprocal of $\frac{1}{5}$, etc.]

✓**Ex. 6.** The difference between a proper fraction and its reciprocal is $\frac{9}{20}$. Find the fraction. [*C. U.* '41 ; *D. B.* '29]

Suppose x is the proper fraction, then its reciprocal is $\frac{1}{x}$.

$\because$ x is a *proper* fraction, $\therefore$ its denominator must be greater than the numerator.

$\therefore$ The numerator of $\frac{1}{x}$ is greater than the denominator. $\therefore$ $\frac{1}{x} > x$.

Now, by the question we have $\frac{1}{x} - x = \frac{9}{20}$, or, $20 - 20x^2 = 9x$,

or, $20x^2 + 9x - 20 = 0$, or, $20x^2 + 25x - 16x - 20 = 0$,
or, $(4x+5)(5x-4) = 0$, $\therefore$ $x = -\frac{5}{4}$ or $\frac{4}{5}$.

Since $-\frac{5}{4}$ is not a proper fraction, it is inadmissible.

$\therefore$ The required fraction is $\frac{4}{5}$.

Ex. 7. The sum of two numbers is 45 and the mean proportional between them is 18 ; find them. [*B. U.* '29]

Let x be one of the numbers. Then the other number is $(45 - x)$.

$\because$ 18 is the mean proportional between x and $(45 - x)$,

$\therefore$ $x(45 - x) = 18^2$, or, $45x - x^2 = 324$, or, $x^2 - 45x + 324 = 0$,

or, $(x - 36)(x - 9) = 0$, $\therefore$ $x = 9$ or 36.

$\therefore$ The required numbers are 9 and 36.

✓**Ex. 8.** A certain number exceeds its reciprocal by 1. How many such numbers are there ? Find them. [*C. U.* '34]

Let x be the number, then its reciprocal is $\frac{1}{x}$.

$\therefore$ From the given condition we have

$$x - \frac{1}{x} = 1, \text{ or, } x^2 - 1 = x, \text{ or, } x^2 - x - 1 = 0,$$

$$\therefore \quad x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times -1}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$\therefore$ there are two such numbers, which are

$$\frac{1 + \sqrt{5}}{2} \text{ and } \frac{1 - \sqrt{5}}{2}.$$

✓Ex. 9. Show that the product of any four consecutive numbers increased by unity is a perfect square.

Let x, $x+1$, $x+2$ and $x+3$ be 4 consecutive numbers. By adding 1 to their product we have

$$x(x+1)(x+2)(x+3)+1=\{x(x+3)\}\{(x+1)(x+2)\}+1$$

$$=(x^2+3x)(x^2+3x+2)+1=a(a+2)+1 \text{ [putting } a \text{ for } x^2+3x]$$

$$=a^2+2a+1=(a+1)^2$$

$$=(x^2+3x+1)^2,$$ which is a perfect square.

Ex. **10.** The hypotenuse of a right-angled triangle is 20 inches. If the difference between its other sides be 4 inches, find the sides. [*G. U.* '49]

Of the two other sides let the smaller one be x inches. Then the other side is $(x+4)$ inches.

$\therefore\ x^2+(x+4)^2=20^2$, or, $x^2+x^2+8x+16=400$,

or, $2x^2+8x-384=0$, or, $x^2+4x-192=0$,

or, $(x+16)(x-12)=0$, $\therefore\ x=-16$ or 12.

$\therefore$ one side $=12$ in., and the other side $=(12+4)$ or 16 inches.

Ex. **11.** The area of a rectangular plot of land fenced all round is 2000 sq. yards and the total length of fencing is 180 yds. Obtain a quadratic equation to determine the length of the plot.

[*C. U.* '33]

Here, 2(length + breadth) = the length of the fencing = 180 yds.

$\therefore$ length + breadth = 90 yds.

Let the length be x yds.; then the breadth $=(90-x)$ yds.

Now, by the question we have $x(90-x)=2000$ sq. yds.

or, $x^2-90x+2000=0$, this is the quadratic equation required.

Again, $x^2-90x+2000=0$, or $x^2-50x-40x+2000=0$,

or, $(x-50)(x-40)=0$, $\therefore\ x=50$ or 40.

$\therefore$ The length of the plot is 50 yds. ($\because$ length > breadth).

Ex. 12. A cyclist travels 84 miles and finds that he could have made the journey in 5 hours less if he had travelled 5 miles an hour faster. At what rate did he travel? [*G. U.* '50]

Let the rate be x miles per hour. At this rate he goes 84 miles in $\frac{84}{x}$ hours. If the rate had been $(x+5)$ miles per hour, the time taken would have been $\frac{84}{x+5}$ hours.

$\therefore$ By the question we have $\frac{84}{x}-\frac{84}{x+5}=5$,

or, $\frac{84x+420-84x}{x(x+5)}=5$, or, $\frac{420}{x^2+5x}=5$, or, $5(x^2+5x)=420$,

or, $x^2+5x=84$, or, $x^2+5x-84=0$, or, $(x+12)(x-7)=0$,

$\therefore$ $x=-12$ or 7.

$\because$ The rate cannot be negative, $\therefore$ the required rate is 7 miles per hour.

Ex. 13. The perpendicular drawn from the centre of a circle to a certain chord is 3 inches less than half the chord. If the radius be 15 inches, find the length of the chord.

The perpendicular drawn from the centre to a chord of a circle bisects the chord.

Let OD be the perpendicular drawn from the centre O to the chord AB of a circle. Then OAD is a right-angled triangle.

Let OD be x inches, then $AD=x+3$ inches and the hypotenuse $OA=15$ inches. $\because$ $OD^2+AD^2=OA^2$,

$\therefore$ $x^2+(x+3)^2=15^2$, or, $2x^2+6x+9=225$,

or, $x^2+3x-108=0$, or, $(x-9)(x+12)=0$, $\therefore$ $x=9$ or -12.

$\because$ The length of the perpendicular cannot be negative,

$\therefore$ the perpendicular $=9$ inches. $\therefore$ The required length of the chord $=2(x+3)$ in. $=2(9+3)$ in. $=24$ inches.

Ex. 14. A boatman can row 7 miles down a river and back in $4\frac{2}{3}$ hours. If the river runs at 2 miles an hour, find the rate of the pull in still water. [*D. B.* '47]

Let the rate of the pull in still water be x miles per hour. Then the boat goes $(x+2)$ miles per hour down the river and $(x-2)$ miles per hour up the river.

$\therefore$ We have $\frac{7}{x+2}+\frac{7}{x-2}=4\frac{2}{3}$, or, $\frac{1}{x+2}+\frac{1}{x-2}=\frac{2}{3}$ (dividing by 7),

or, $\frac{x-2+x+2}{x^2-4}=\frac{2}{3}$, or, $\frac{2x}{x^2-4}=\frac{2}{3}$,

or, $x^2-3x-4=0$, or, $(x-4)(x+1)=0$, $\therefore x=4$ or -1.

$\because$ The rate of the pull cannot be negative,

$\therefore$ the required rate is 4 miles per hour.

Ex. 15. A regiment of soldiers, when formed into a solid square, has 16 men fewer in the front than when formed into a hollow square 4 deep. Find the number of soldiers. (*D. B.* '40]

Let x be the number of soldiers in the front, when they are formed into a hollow square.
Then the total number of soldiers $=x^2-(x-4\times 2)^2=x^2-(x-8)^2$.

Again, when formed into a solid square, the number of soldiers in the front is $x-16$. $\therefore$ The total number of soldiers $=(x-16)^2$.

$\therefore$ By the question we have $(x-16)^2=x^2-(x-8)^2$,

or, $x^2-32x+256=x^2-x^2+16x-64$, or, $x^2-48x+320=0$,

or, $(x-40)(x-8)=0$, $\therefore x=40$ or 8.

Here, evidently the value 8 does not satisfy the conditions.

$\therefore$ The required number of soldiers $=(x-16)^2=(40-16)^2$

$$=24^2=576.$$

Ex. 16. What is the price of eggs per dozen, if one more for 6 annas reduces the price by one anna per dozen?

[*D.B.* '39, '41, '46]

Suppose the price of x eggs $=6$ annas,

$\therefore$ " " 12 eggs $=\frac{72}{x}$ annas...(1)

Now, if the price of $(x+1)$ eggs be 6 annas,

the price of 12 eggs is $\frac{72}{x+1}$ annas...(2)

$\therefore$ By the problem, $\frac{72}{x}-\frac{72}{x+1}=1$, or, $\frac{72x+72-72x}{x(x+1)}=1$,

or, $x^2+x=72$, or, $x^2+x-72=0$,

or, $(x+9)(x-8)=0$, $\therefore$ $x=-9$ or 8.

$\because$ The number of eggs cannot be negative,

$\therefore$ the number of eggs $=8$.

Now, from (1), the price per dozen $=\frac{72}{8}$ or 9 annas.

Ex. 17. Find the fraction which assumes twice or thrice its original value when 2 or 3 respectively is added to both its numerator and denominator. [*D. B.* '49]

Let $\frac{x}{y}$ be the fraction. Now, from the first condition $\frac{x+2}{y+2}=\frac{2x}{y}$,

or, $xy+2y=2xy+4x$, or, $2y-4x=xy$,

or, $\frac{2y}{xy}-\frac{4x}{xy}=1$, or, $\frac{2}{x}-\frac{4}{y}=1$...(1)

From the second condition, $\frac{x+3}{y+3}=\frac{3x}{y}$, or, $3y-9x=2xy$,

or, $\frac{3}{x}-\frac{9}{y}=2$......(2).

Now, multiplying (1) by 3 and (2) by 2 we have

$$\frac{6}{x}-\frac{12}{y}=3$$

$$\text{and}\quad \frac{6}{x}-\frac{18}{y}=4$$

(Subtracting) $\frac{6}{y}=-1$, $\therefore$ $y=-6$.

Putting $y=-6$ in (1) we have $\frac{2}{x}+\frac{4}{6}=1$, or, $\frac{2}{x}=\frac{1}{3}$, $\therefore$ $x=6$.

$\therefore$ The reqd. fraction$=\frac{6}{-6}$.

[**N. B.** Here the answer is $\frac{6}{-6}$ and not $\frac{1}{-1}$ or -1, as in the latter case the conditions will not be satisfied.]

Ex. 18. The product of two numbers is 24. If the sum of the squares of the two numbers be added to the sum of the numbers, the result is 62. What are the numbers ?

[*E. B. S. B.* '53]

Let x and y be the numbers.

$\therefore$ $xy=24$...(1) and $x^2+y^2+x+y=62$...(2).

From (2) we have $(x+y)^2-2xy+x+y=62$,

or, $(x+y)^2-2\times24+(x+y)=62$,

or, $(x+y)^2+(x+y)-110=0$,

or, $a^2+a-110=0$ [putting a for $x+y$],

or, $(a+11)(a-10)=0$, $\therefore$ $a=-11$ or 10.

(i) If $a=-11$, then $x+y=-11$.

$\therefore$ $(x-y)^2=(x+y)^2-4xy=(-11)^2-4\times24=25$,

$\therefore$ $x-y=\pm5$.

$$\begin{aligned}\text{Now, } x+y&=-11\\ x-y&=5 \text{ [when } x-y=5]\\ \hline\end{aligned}$$

$\therefore$ $2x=-6$, $\therefore$ $x=-3$, so $y=-8$.

$\therefore$ The required numbers$=-3, -8$.

[The same two numbers are obtained when $x-y=-5$.]

(ii) Again, if $a=10$, then $x+y=10$, $\therefore$ $y=10-x$.

$\because$ $xy=24$, $\therefore$ $x(10-x)=24$,

or, $x^2-10x+24=0$, or, $(x-4)(x-6)=0$,

$\therefore$ $x=4$, or, 6 ; Hence $y=6$ or 4.

$\therefore$ The required numbers$=4, 6$.

Thus from (i) and (ii) we have,

the reqd. numbers$=4$ and 6, or, -3 and -8.

Exercise 11

1. Find a number such that its square added to its cube is 16 times the next number. [*A. U.* '16]

2. The sum of two numbers is 2 and the sum of their reciprocals is $2\frac{1}{4}$. Find the numbers. [*C. U.* '35]

3. Find two consecutive odd numbers whose product is 35.

4. Find two consecutive numbers such that the difference of their reciprocals is $\frac{1}{110}$.

5. What number when added to 30 will be less than its square by 12 ? [*E.B.S.B.* '50]

6. The sum of the squares of two consecutive odd integers is 290 ; find the integers.

7. A number is greater than its square root by 110. Find it.

8. The sum of the squares of two consecutive even numbers is 100. Find the numbers. [*A.U.* '24]

9. In a right-angled triangle the sum of the two sides containing the right angle is 23″, and the hypotenuse is 17″. Find the sides containing the right angle. [*G. U.* '51]

10. The hypotenuse of a right-angled triangle is 13 inches. Find the length of each of the remaining two sides if their sum is 17 inches. [*C. U.* '45]

11. A and B can do a piece of work in 72 mins., but B alone takes 1 hr. more than A to do it. In what time can each do it ?

✓**12.** Find the price of eggs per dozen when two more eggs for a shilling would reduce the price by one penny per dozen. [*E. B. B. S.* '51]

13. A man bought a certain number of goats for Rs. 420 ; had he obtained one more for the same sum, the price of each would have been Re. 1 less. Find the number of goats bought.

✓**14.** A number is less than twice the product of its two digits by 8 ; if the digit in the tens' place is greater than the digit in the units' place by 1, find the number.

15. A man takes one hour less to ride 24 miles if he increases his speed by 2 miles per hour. Find his speed per hour.

16. The perpendicular from the centre of a circle to a chord is less than half the chord by 1″ ; if the radius be 5″, find the length of the chord.

✓**17.** The area of a rectangular field is 260 sq. ft. If its length be diminished by 5 ft. and breadth increased by 2 ft., it becomes a square field. Find its length and breadth.

18. A boatman can row 9 miles down the river but 3 miles up stream in a certain time. If the speed of the current be 2 miles per hour, find the speed of the boat in still water.

19. Find two consecutive positive numbers the sum of whose squares is 761. [*G.U.* '52]

20. The hypotenuse of a right-angled triangle is 25″. If the sum of the three sides be 56″, find the smallest side of the triangle. [*G.U.* '54]

21. The circumference of the fore-wheel of a carriage is 4 metres less than that of the hind wheel and it makes 15 revolutions more than the latter in 240 metres. Find the circumference of each wheel.

22. A company of soldiers, when formed into a solid square, has 12 men fewer in the front than when formed into a hollow square 3 deep. Find the number of soldiers.

GRAPH

You have already learnt the method of drawing graphs.

54. The following facts regarding graphs should be carefully noted :

1. The graph of any simple equation connecting two variables x and y in the first degree is a straight line. It is a **linear** graph. Here the equation is of the form $y=mx$ or $y=mx+c$.

2. The graph of a quadratic equation, which can be expressed in the form $XY=0$ (*i.e.*, can be reduced into two simple equations $X=0$ and $Y=0$) consists of a **pair of straight lines.**

Thus (*i*) the graph of $x^2-x-6=0$, or, $(x-3)(x+2)=0$, consists of a pair of graphs of the equations $x-3=0$ and $x+2=0$.

(*ii*) The graph of $x^2=9y^2$, or $x^2-9y^2=0$, or $(x+3y)(x-3y)=0$ will consist of two st. lines representing $x+3y=0$ and $x-3y=0$. When a quadratic equation cannot be thus reduced to the form $XY=0$, its graph is non-linear and is said to be a **conic.**

3. If in a quadratic equation, the coefficients of x^2 and y^2 are equal and positive and if there is no term involving xy, its graph will be a **circle.** The equation of a circle may be of the forms : (*i*) $x^2+y^2=a^2$, (*ii*) $(x+b)^2+(y+c)^2=a^2$, and (*iii*) $x^2+y^2+ax+by+c=0$.

Thus (*i*) $x^2+y^2=16$, (*ii*) $(x+2)^2+(y-3)^2=9$, (*iii*) $x^2+y^2+12x+18y+92=0$ all represent circles.

4. If in a quadratic equation, one of the two variables x and y is of the second degree and the other of the first degree and if there is no term involving xy, its graph will be a **parabola.**

Such equations are of the forms $y=mx^2$, or, $mx=y^2$. The graph of an equation of the form $y=ax^2+bx+c$ or $x=ay^2+by+c$ will be a parabola, if $a\neq 0$. Thus (*i*) $4y=x^2$, (*ii*) $x=y^2$ and (*iii*) $y=x^2+2x+1$ represent parabolas.

5. The graphs of the **functions** $4x+2, x^2, x^2-5x+6$, etc. are the same as the graphs of the equations $y=4x+2$, $y=x^2$, $y=x^2-5x+6$ respectively. So, in drawing graphs of such a function, put the given expression$=y$ and draw the graph of that equation.

6. The graph of the quadratic equation, in which the coefficients of x^2 and y^2 are unequal and positive, is an **ellipse.**

Such equations are of the form $ax^2+by^2=c^2$.

Thus $4x^2+9y^2=25$ represents an **ellipse.**

7. The graph of the quadratic equation, in which the coefficients of x^2 and y^2 are of opposite signs (one positive, the other negative), is a **Hyperbola.** Such equations are of the form $ax^2-by^2=c^2$. Thus $9x^2-25y^2=225$ represents a hyperbola.

8. The graph of an equation in the form $xy=a$, or $xy+ax+by=c$ is a particular hyperbola, known as **Rectangular Hyperbola.** Thus $xy=8$ represents such a hyperbola.

The origin is called the centre of the rectangular hyperbola and its axes are the bisectors of the angles between the axes of co-ordinates.

9. To solve a quadradic equation graphically, any one of the following methods may be adopted :

To solve $x^2-7x+22=0$, we may either (*i*) put $y=x^2-7x+22$ and $y=0$, or, (*ii*) we may write the equation as $x^2=7x-22$, and then put $y=x^2$ and $y=7x-22$.

Now, if either in (*i*) or in (*ii*) the graphs of the two equations in each be traced, the abscissæ of the points of intersection of the graphs will give the required roots of the given equation.

10. In drawing a graph the axes of reference, the origin and the unit used must be stated in each case.

55. Drawing of graphs and graphical solutions of equations.

Examples 14

Ex. 1. Solve graphically $3x=17-2y$ and $3y=2x+6$. [*A.U.*'27]

We have $3x=17-2y$......(1), or, $2y=17-3x$,

$\therefore \quad y=\dfrac{17-3x}{2}$; then

x	-1	3	5
y	10	4	1

Again, $3y=2x+6$......(2), or, $y=\dfrac{2x+6}{3}$,

then

x	0	-3	6
y	2	0	6

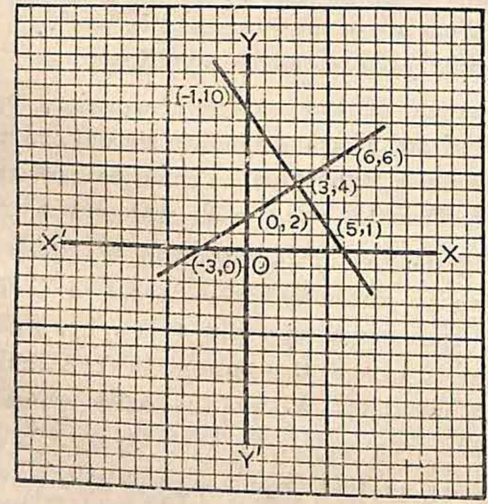


Fig. 1

Let XOX' and YOY' be the axes and O the origin. Draw the graphs, taking one side of a small square on the graph paper as unit of length. The two graphs intersect at a point whose co-ordinates found from the graph are (3, 4).

$\therefore$ The required solution is $x=3, y=4$.

Ex. 2. Solve graphically $\frac{8-x}{2}=\frac{x-3}{3}$.

Let $y=\frac{8-x}{2}$, then $y=\frac{x-3}{3}$.

From $y=\frac{8-x}{2}$ we have

x	0	2	4
y	4	3	2

And from $y=\frac{x-3}{3}$ we have

x	0	3	-3
y	-1	0	-2

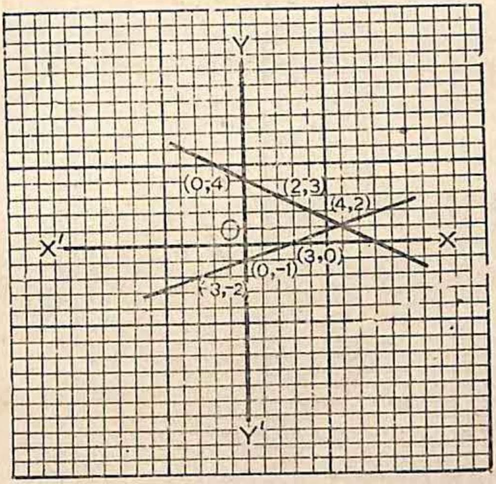


Fig. 2

Draw the two graphs, taking one side of a small square on the graph paper as the unit of length. The graphs intersect at the point (6, 1). [Fig. no. 2]. $\therefore$ The solution is $x=6$.

Ex. 3. Solve graphically $x-3=2$.

Here the abscissa of the point of intersection of the graphs of $x-3$ and 2 is the root of x.

Let $y=x-3$, then $y=2$. The graphs of these two equations are the graphs of $x-3$ and 2 respectively.

From $y=x-3$ we have

x	2	3	5
y	-1	0	2

;

and from $y=2$,, ,,

x	0	1	-3
y	2	2	2

;

The two graphs are drawn taking one side of a small square on the graph paper as the unit of length. They intersect at a point whose co-ordinates are (5, 2). $\therefore$ The solution is $x=5$.

Ex. 4. Draw the graph of $y-2x+4=0$ and find from it the solution of the equation $2x-4=0$. [*D. B.* 1929]

Here $y-2x+4=0$, $\therefore$ $y=2x-4$...(A)

From (A) we have

x	0	-1	3
y	-4	-6	2

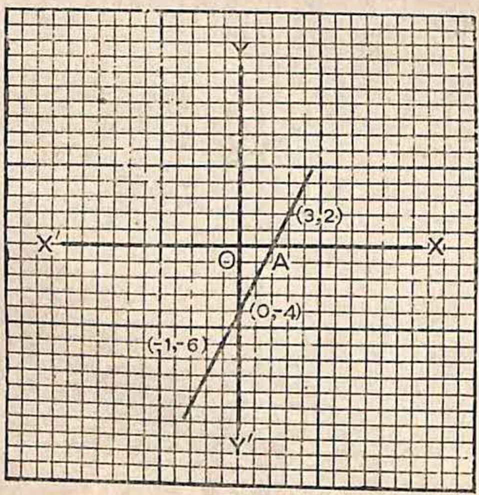


Fig. 3

Let the graph be drawn taking one side of a small square on the graph paper as the unit of length (fig. 3).

Now, to solve the equation $2x-4=0$ from the graph, we have to find the abscissa of the point A where the graph cuts the x-axis, because the ordinate of A is 0 (*i.e.* $y=0$) and hence we have $2x-4=0$, *i.e.* equation (A)

$\because$ the abscissa of A is 2 units, $\therefore$ the required solution is $x=2$.

Graphs of a pair of straight lines

Vide the notes in Art. 54(2) in this connection.

Ex. 1. Draw the graph of $x^2-x-6=0$.

Here $x^2-x-6=(x-3)(x+2)$.

$\therefore$ We have $(x-3)(x+2)=0$, so that $x-3=0$ and $x+2=0$.

So the required graph of the given quadratic equation will be a pair of straight lines representing $x-3=0$......(i)

and $x+2=0$......(ii)

Now draw the graphs of (i) and (ii) taking one small division on the graph paper as the unit of length. The graph (i) is a straight line parallel to the y-axis at a distance of 3 units from it and that of (ii) is a straight line parallel to y-axis at a distance of -2 units from it.

The required graph is the pair of these two straight lines.

Ex. 2. Draw the graph of $6x^2+5xy-6y^2-12x-18y=0$.

Here $6x^2+5xy-6y^2-12x-18y=(2x+3y)(3x-2y-6)$.

$\therefore$ We have $(2x+3y)(3x-2y-6)=0$.

$\therefore$ $2x+3y=0$......(i) and $3x-2y-6=0$......(ii)

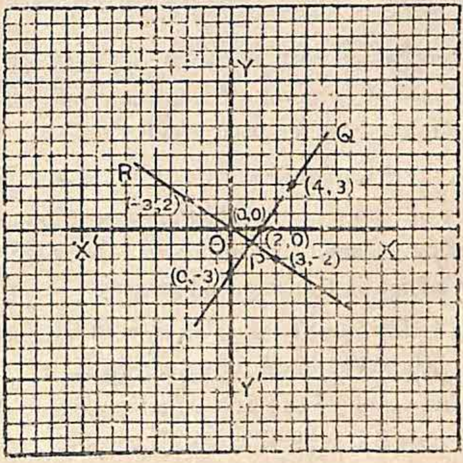


Fig. 4

So the graph of the given equation will be a pair of straight lines representing (i) and (ii). Draw the graphs of (i) and (ii) taking one small division on the graph paper as the unit of length. The required graph is the pair of the two straight lines drawn.

[Circle]

Ex. 1. Draw the graph of $x^2+y^2=16$.

Let XOX$'$ and YOY$'$ be the axes and O the origin.

The given equation may be written as $\sqrt{x^2+y^2}=\sqrt{16}=4$.

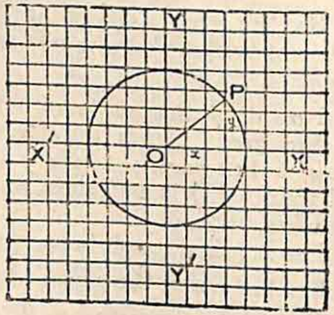


Fig. 5

Let P be any point on the graph and (x, y) be the co-ordinates of P. Now, the distance of P from the origin O is OP. Here $OP^2=x^2+y^2$,

$\therefore$ $OP=\sqrt{x^2+y^2}=4$(constant).

$\therefore$ The distance of P from the fixed origin is always 4 units. Hence the locus of the point P is a circle whose centre is O and radius is 4 units of length. Let one side of a small square on the graph paper be the unit of length.

Let a circle be drawn with centre O and with radius $=4$ units. As the co-ordinates of no point outside the circle satisfy the given equation, this circle is the required graph.

[**Otherwise :** $\because$ $x^2+y^2=16$, $\therefore$ $y=\pm\sqrt{16-x^2}$. Now the co-ordinates of the points on the graph can be tabulated from the above equation. Plot those points and draw through them a freehand curve, which is the required graph.]

Ex. 2. Draw the graphs of $x^2+y^2=16$ and $x+y=2$, and measure the length of the chord of intersection. [*C. U.* '13]

[*Hints* :—The graph of the first equation is a circle (See Ex.1). The graph of $x+y=2$ is a st. line drawn through the points $(1, 1)$, $(5, -3)$, $(-4, 6)$. Let the graphs be drawn taking one small division on the graph paper as the unit of length.

Let the graphs intersect at A and B.

Now from OX cut off OP$=$AB. Here OP$=7{\cdot}5$ units.

$\therefore$ The length of the chord of intersection$=7{\cdot}5$ units].

Ex. 3. Draw the graph of $(x-2)^2+(y-3)^2=25$.

From the given equation we have $\sqrt{(x-2)^2+(y-3)^2}=5$.

Plot a point A whose co-ordinates are (2, 3), taking one side of a small square on the graph paper as the unit of length.

Let (x, y) be the co-ordinates of any point P on the required graph. Then $AP^2=(x-2)^2+(y-3)^2$.

$\therefore AP=\sqrt{(x-2)^2+(y-3)^2}$
$=5$ (constant).

$\therefore$ P is always at a distance of 5 units from A and so its locus is a circle.

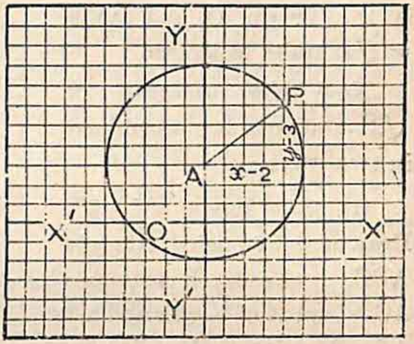


Fig. 6

$\therefore$ The circle drawn with centre A and with radius = 5 units of length is the required graph. Draw the graph with the unit of length mentioned above.

[**N. B.** As we have $(x-2)^2$ and $(y-3)^2$ in the given equation, we take (2, 3) as the co-ordinates of A (i. e., the co-ordinates are of the opposite signs of -2, -3).

If $(x+3)^2+(y-4)^2=36$ be the given equation, we should first plot the pt. $(-3, 4)$ and take it as the centre of the circle and 6 units of length as its radius.]

Ex. 4. Draw the graph of $x^2+y^2+12x+18y+92=0$.

Here, $x^2+y^2+12x+18y+92=0$,

or, $(x^2+12x+36)+(y^2+18y+81)-25=0$,

or, $(x+6)^2+(y+9)^2=25=5^2$.

$\therefore$ The graph of the equation is a circle whose centre is the point $(-6, -9)$ and radius = 5 units of length. Let the circle be drawn taking one side of a small square as the unit of length.

Ex. 5. Draw the graph of $x^2+y^2=21$.

$$x^2+y^2=21, \quad \text{or,} \quad \sqrt{x^2+y^2}=\sqrt{21}.$$

Evidently the graph of the equation is a circle, whose centre is the origin O and radius $=\sqrt{21}$ units of length.

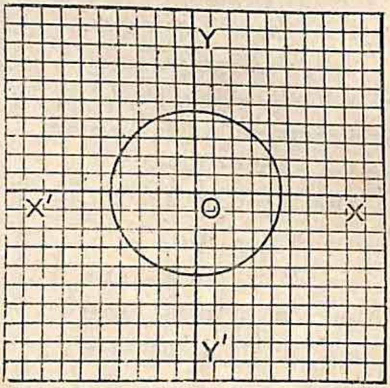


Fig. 7

Now the measure of $\sqrt{21}$ units can be determined in two ways :

(*i*) $21=25-4=5^2-2^2$. Draw a right-angled triangle ABC, whose hypotenuse AC=5 units of length and base AB=2 units. [Fig. 7 (A)] Then, $\because$ $BC^2=AC^2-AB^2$
$=5^2-2^2=21$,

$\therefore$ $BC=\sqrt{21}$ units.

(*ii*) $21=7\times3$; take a st. line $PQ=7+3=10$ units of length. From PQ cut off $PR=7$ units of length, then $RQ=3$ units. Draw a circle on PQ as diameter and draw $RD\perp PQ$ and let RD cut the semi-circle at D. Then $RD=\sqrt{21}$ units of length. [Fig. 7(A)]

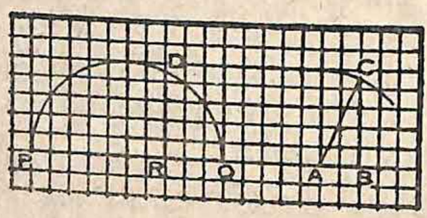


Fig. 7(A)

Now, draw a circle with centre O and with radius equal to BC or RD. This circle is the required graph.

Ex. 6. Draw the graphs of $x^2+y^2=25$ and $\frac{x}{5}+\frac{y}{5}=1$, and find the co-ordinates of their points of intersection graphically.
[*C. U.* '39 Sup., '43]

From the equation $x^2+y^2=25$, we have $\sqrt{x^2+y^2}=5$.

Now, $\sqrt{x^2+y^2}$ denotes the distance of a point $P(x, y)$ from the origin O. Since $\sqrt{x^2+y^2}=5$ (a constant), P is always equidistant from O and so its locus is a circle.

Let axes XOX′ and YOY′ intersect at right angles at O. Take one small division on the graph paper as the unit of length and draw a circle with centre O and with radius equal to 5 units of length. This circle is the graph of the first equation.

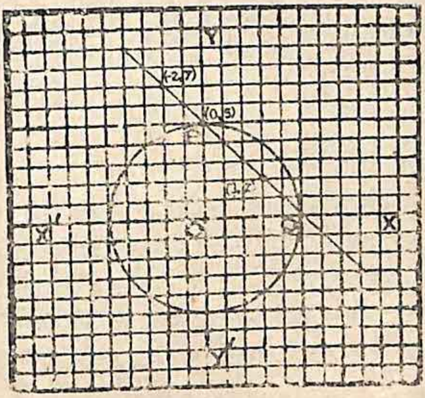


Fig. 8

Again, from $\frac{x}{5}+\frac{y}{5}=1$, we have $x+y=5$, then $\begin{array}{c|c|c|c} x & 3 & -2 & 0 \\ \hline y & 2 & 7 & 5 \end{array}$.

Evidently the graph of the second equation is a straight line drawn through the points $(3, 2)$, $(-2, 7)$, $(0, 5)$. Draw this graph with the same unit as before.

The graphs intersect each other at P and Q, whose co-ordinates found from the graph are $(0, 5)$ and $(5, 0)$ respectively.

Ex. 7. Draw the graph of $x^2+y^2=25$ and show that $3x+4y=25$ touches it at (3, 4). [*C. U.* '11 ; *E. B. S. B.* '50]

The graph of the first equation is a circle. Draw the graph as in Ex. 1.

From the equation $3x+4y=25$, we have $\begin{array}{c|c|c|c} x & 3 & 7 & -1 \\ \hline y & 4 & 1 & 7 \end{array}$;

the straight line AB drawn through the points (3, 4), (7, 1), (− 1, 7) and produced both ways is the graph of the second equation.

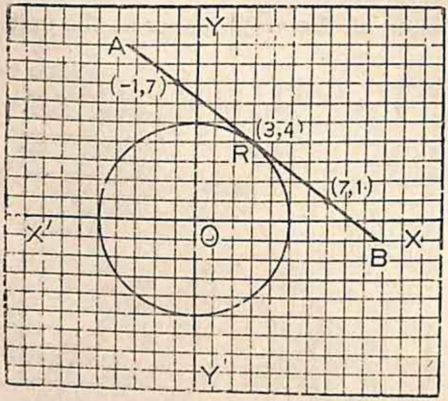


Fig. 9

Draw the graphs taking each small division on the graph paper as the unit of length.

Here the straight line meets the circle only at one point R. ∴ it touches the circle at R, whose co-ordinates are (3, 4).

[**Geometrical Proof** : ∵ The co-ordinates (3, 4) satisfy both the equations, ∴ both the graphs meet at the pt. (3, 4). **From** RO cut off RC = 3 units and from RA cut off RD = 4 units of length. By measurement CD is found to be 5 units in length.

$$\because \quad RC^2+RD^2=3^2+4^2=25=5^2=CD^2,$$

∴ ∠R is a right angle. ∴ AB, being perpendicular to the radius OR at R, touches the circle at R.]

Ex. 8. Solve graphically the simultaneous equations (i) $x^2+y^2=53$ and (ii) $y-x=5$.

The graph of $x^2+y^2=53$ is a circle. Now $53=2^2+7^2$, so $x=2$ and $y=7$ satisfy the equation.

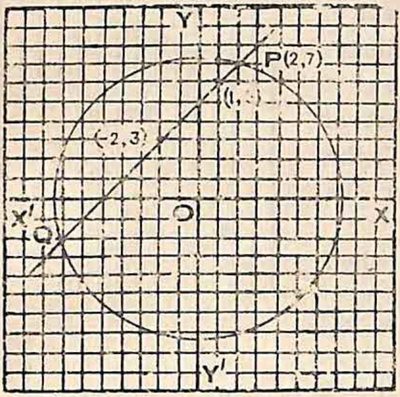


Fig. 10

Plot the point P(2, 7) on the graph paper, taking each small division on it as the unit of length. Then P is a point on the graph. Draw a circle with centre O and with radius OP, which is the graph of the first equation.

(ii) From $y-x=5$, we have $y=5+x$,

x	-2	1	2
y	3	6	7

Draw the graph with the same unit as before. The two graphs intersect at P and Q whose co-ordinates are found to be (2, 7) and (−7, −2) respectively.

$\therefore$ the required solution is $x=2$ and $y=7$,

or, $x=-7$ and $y=-2$.

[Parabola]

Ex. 1. Draw the graph of $y=x^2$ between the limits $x=3$ $x=-3$. and hence find the value of $\sqrt{5}$ to the first decimal place. [*C. U.* '29, '38]

The graph of $y=x^2$ is a parabola. From $y=x^2$ we have

x	0	±1	±2	±3	$\pm\cdot5$	$\pm1\cdot5$	$\pm2\cdot5$	...
y	0	1	4	9	$\cdot25$	$2\cdot25$	$6\cdot25$	...

The above points (0, 0), (1, 1), (−1, 1), (2, 4), (−2, 4), etc. are on the graph.

Let the axes XOX′, YOY′ intersect at right angles at O. Plot the above points, taking one small division on the graph paper as the unit of length. Draw a **continuous freehand curve** through these points. This curve which is a **parabola** is the required graph.

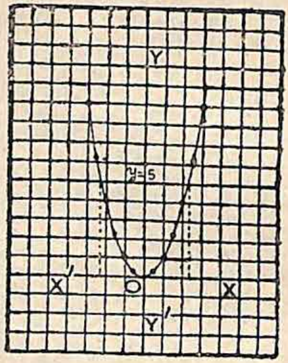


Fig. 11

Now, we have to find the value of $\sqrt{5}$ from this graph. The co-ordinates of each of the above points show that the value of x is the square root of the value of y. Hence if $y=5$, the corresponding value of x will be the value of $\sqrt{5}$.

From the graph we find that when $y=5$, the values of $x=\pm 2{\cdot}2$.

$\therefore \quad \sqrt{5} \equiv \pm 2{\cdot}2$ (App.).

N. B. The students should observe the following points :

(1) From the equation we have $x=\pm\sqrt{y}$. Hence for every value of y (the ordinate), x(the abscissa) has two values, equal but opposite in sign. So the graph is symmetrical with respect to the axis of y, *i.e.*, the part of the graph on either side of this axis is an exact image of the other part.

(2) Since the value of y must always be positive, whatever positive or negative value x may have, all the plotted points must lie above the axis of x.

(3) The graph passes through the origin.

(4) As the mumercial values of x increase, those of y increase rapidly. Since there is no limit to the values which x may have, both parts of the graph extend upwards and outwards to infinity. This is indicated by extending both the parts of the graph symmetrically a little beyond the last plotted points. In the above graph, however, it has not been so extended, as there are limits to the values of x.

(5) In drawing a parabola at least six points should be plotted.

Ex. 2. Draw the graphs of $y^2 = x$ and $y = x$, and find the co-ordinates of their points of intersection. [*G. U.* '50]

From $y^2 = x$, we have

x	0	1	4	9	·25	2·25	...
y	0	± 1	± 2	± 3	$\pm$·5	$\pm$1·5	...

From $y = x$, whe have

x	0	2	-3	...
y	0	2	-3	...

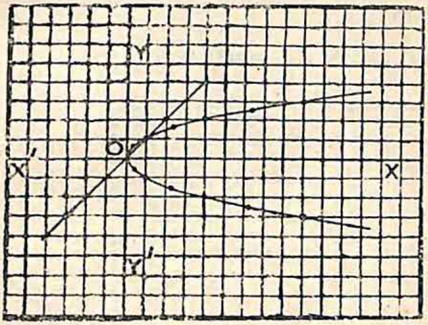


Fig. 12

The two graphs are drawn, taking one side of a small square on the graph paper as the unit of length. The graph of the first equation is a parabola and that of the second is a straight line. They intersect at two points, whose co-ordinates are found to be (0, 0) and (1, 1).

Graphical Solutions of Quadratic Equations

Ex. 3. Draw the graph of $y = 2x - \frac{x^2}{4}$ and obtain the roots of the equation $2x - \frac{x^2}{4} = 0$ from the graph.

From the equation $y = 2x - \frac{x^2}{4}$ we have

x	0	1	-1	2	-2	3	-3	4	5	6
$2x$	0	2	-2	4	-4	6	-6	8	10	12
$-\frac{x^2}{4}$	0	$-$·25	$-$·25	-1	-1	$-$2·25	$-$2·25	-4	$-$6·25	-9
y	0	1·75	$-$2·25	3	-5	3·75	$-$8·25	4	3·75	3

The graph is drawn plotting the above points with two small divisions on the graph paper as the unit of length. [Vide fig. 13].

Putting 0 for y in the given equation we have $2x - \frac{x^2}{4} = 0$. Hence to solve the equation $2x - \frac{x^2}{4} = 0$ from the graph, we have to find the abscissa of the point, where the graph intersects the x-axis. Here the graph cuts the x-axis at O and P. The abscissa of the pt. O is zero and that of P is 8.

∴ The required roots are 0 and 8. *i.e.*, $x = 0$ or 8.

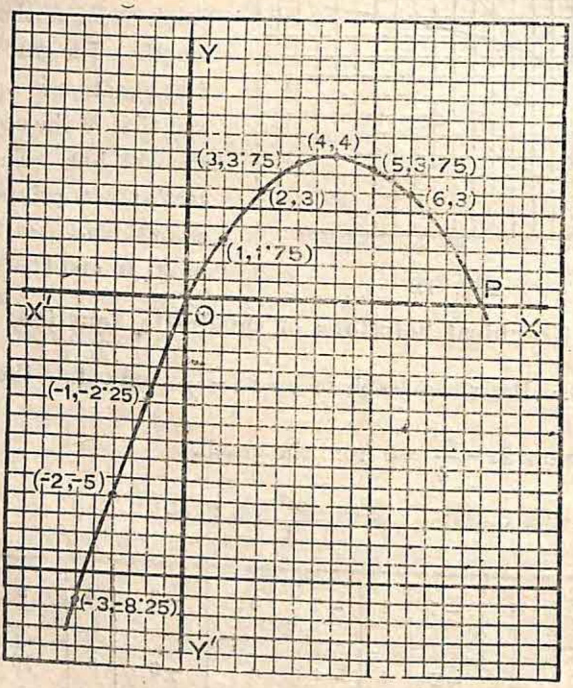


Fig. 13

Ex. 4. Give the graphical solution of $2x^2 - x - 2 = 0$. [*A.U.*'15]

Here $2x^2 - x - 2 = 0$, or, $2x^2 = x + 2$, $\therefore \; x^2 = \frac{x+2}{2}$;

Let $x^2 = y$...(1) ; then $y = \frac{x+2}{2}$......(2). From (2) we have

x	0	2	-2
y	1	2	0

Now, draw the graphs of the two equations. The first graph is a parabola (Vide Ex. 1) and the second graph is a straight line. Now the abscissæ of the two points where the two graphs intersect are 1·3 and $-$·8.

∴ The solution is $x = 1·3$ or $-·8$.

[**N. B.** We could solve it by drawing the graphs of $y = 2x^2 - x - 2$ and $y = 0$. The graph of $y = 0$ is the x-axis. The abscissæ of the points of intersection of the graphs would be 1·3 and $-$·8]

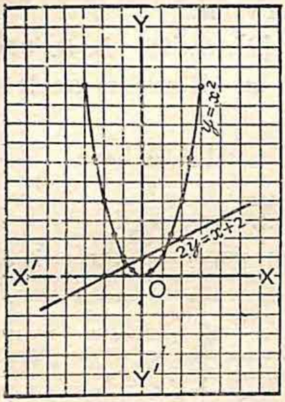


Fig. 14

Ex. 5. Prove graphically that the expression $x^2 - 3x + 4$ is positive for all real values of x. [*C. U.* '48]

Here $x^2 - 3x + 4$ is a function. Its graph is the same as the graph of $y = x^2 - 3x + 4$.

From this equation we have

x	0	1	-1	2	3	4	...
y	4	2	8	2	4	8	...

Plot the above points, taking one side of a small square as the unit of length and draw through them a continuous free hand curve. This curve, which is a parabola. is the graph of the expression $x^2 - 3x + 4$.

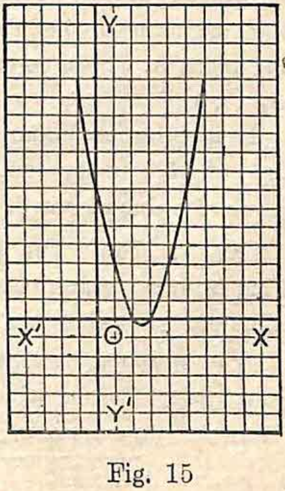


Fig. 15

In the graph we find that no part of it is on the negative side of **y-axis**, *i.e.*, below the **x-axis**. Hence the value of y cannot be negative for any real value of x, positive or negative. Hence the expression is positive for all real values of x.

Ex. 6. Find graphically the maximum value of $6x - x^2 - 5$, and the value of x that gives the maximum value.

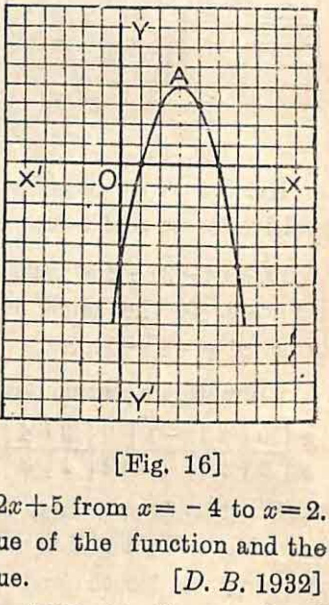
[*M. U.* '28 ; *D. B.* '31]

Let the given function be equal to y. Hence its graph is the same as the graph of $y = 6x - x^2 - 5$.

Tabulating the corresponding values of x and y we have

x	0	1	2	3	5	6	...
y	-5	0	3	4	0	-5	...

Let one small division on the graph paper be the unit of length and plot the above points. Draw through them a continuous freehand curve, which is the graph of the function.



[Fig. 16]

Now, the maximum value of y will be the required maximum value of $6x - x^2 - 5$.

In the graph we find that the ordinate is the greatest at the point A, whose co-ordinates are (3, 4).

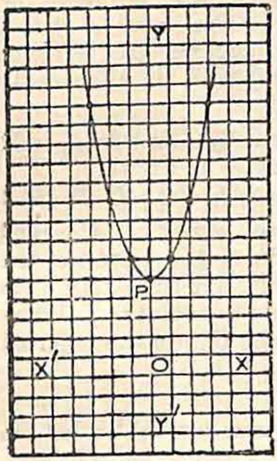
Hence the maximum value of $6x - x^2 - 5$ is 4, when the required value of x is 3.

Ex. 7. Draw the graph of $x^2 + 2x + 5$ from $x = -4$ to $x = 2$. Find from the graph the minimum value of the function and the value of x that gives the minimum value. [*D. B.* 1932]

The graph of $x^2 + 2x + 5$ is the same as the graph of $y = x^2 + 2x + 5$, from which we have,

x	0	1	-1	2	-2	-3	-4	...
y	5	8	4	13	5	8	13	...

Let one small division on the graph paper be the unit of length and plot the above points. Then the parabola obtained by drawing a free-hand curve through the points is the required graph.



[Fig. 17]

The minimum value of y will be the minimum value of the function. In the graph we find that there is no part of it below the point P.

The co-ordinates of P are $(-1, 4)$. $\therefore$ Its ordinate (y) 4 is the minimum value of y, *i.e.*, of the given function, and the value of x that gives this value is -1.

Ex. 8. Trace the changes in sign and magnitude of the expression x^2-3x+2 as x increases from $-\infty$ to $+\infty$. [*C.U.*'49]

The graph of x^2-3x+2 is the same as the graph of $y=x^2-3x+2$ and it is a parabola. From the equation we have

x	0	1	2	3	4	-1	-2	...
y	2	0	0	2	6	6	12	...

Let the axes XOX', YOY' intersect at right angles at the origin O. Let one side of a small square on the graph paper represent the unit of length. Plot the points [Fig. 18] and through them draw a freehand curve, which is the graph of the expression x^2-3x+2.

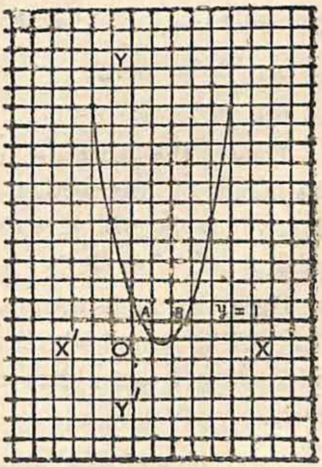


Fig. 18

Now from the graph we find that it may be extended upwards to infinity and that the portion of it below the **x-axis** is within the limits of $x=1$ and $x=2$. So, if the value of x lie between 1 and 2, the value of y (*i.e.*, of the function) is negative. Beyond this limit y will no more be negative, whatever value, positive or negative, may x have. As the value of x (positive or negative) increases gradually to infinity the value of y (*i.e.*, the function) will simultaneously increase to infinity but will always be positive.

Ex. 9. Draw the graphs of $y=x^2$ and $2y-5x+3=0$, and hence obtain the solution of the equation $2x^2-5x+3=0$.

[*W. B. S. F.* '52 ; *D. B.* '29]

[**Hints** : The given equations are $y=x^2$......(1) and $2y-5x+3=0$, or, $y=\frac{5x-3}{2}$...(2). First draw the graphs of the two equations. Now, the equation to be solved is $2x^2-5x+3=0$ or $x^2=\frac{5x-3}{2}$. Evidently the abscissæ of the points of intersection of the graphs of equations (1) and (2) are the required roots. Here the abscissæ are 1 and $1{\cdot}5$.

$\therefore$ The solution is $x=1$ or $1{\cdot}5$.]

Ex. 10. Using the graphs of $y=x^2$ and $y=\frac{5x+2}{3}$ solve the equation $3x^2-5x-2=0$.

[*C. U.* '48]

The graph of $y=x^2$ is a parabola. Draw the graph as in Ex. 1.

From $y=\dfrac{5x+2}{3}$, we have

x	-1	2	-4	...
y	-1	4	-6	...

The st. line drawn through the above points is the graph of the second equation.

$\because$ Both the equations are equal to y, $\therefore$ $x^2=\frac{5x+2}{3}$

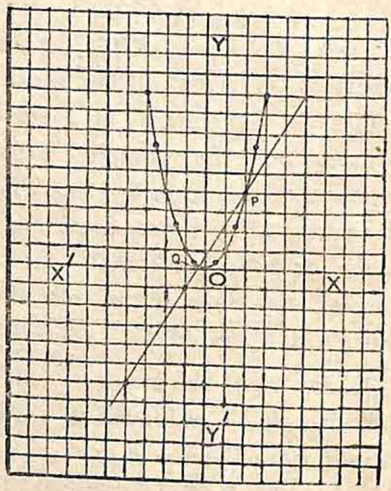
or, $3x^2=5x+2$,

or, $3x^2-5x-2=0$.

Hence the abscissæ of the points of intersection of the two graphs will give the roots of $3x^2-5x-2=0$.

Here, the graphs intersect at P and Q whose abscissæ are 2 and $-{\cdot}3$ (appx.) respectively.

$\therefore$ The required solution is $x=2$ or $-{\cdot}3$.



[Fig. 19]

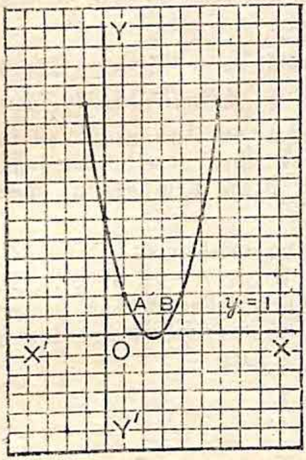
Ex. 11. Draw the graph of $y=x^2-3x+2$ and hence solve the equation $x^2-3x+1=0$.

The graph drawn in Ex. 8 is the graph of $y=x^2-3x+2$ here. We have to solve the equation $x^2-3x+1=0$ from the graph.

We have $x^2-3x+1=0$, or, $x^2-3x+2=1$.

Let $y=x^2-3x+2$, then $y=1$. Then the abscissæ of the points of intersection of the graphs of these two equations will give the roots of $x^2-3x+1=0$.

Draw the two graphs. The graph of $y=1$ is a st. line parallel to the x-axis at a distance of 1 unit from it. The graphs intersect at A and B, whose co-ordinates are $(\cdot 4, 1)$ and $(2\cdot 6, 1)$.



[Fig. 20]

$\therefore$ The required solution is $x=\cdot 4$ or $2\cdot 6$.

Ex. 12. Draw the graph of $y=5x^2+2x-1$. Read off the abscissa where this graph is cut by $y=3$. These are the roots of the equation $5x^2+2x-4=0$. Explain why. [*C. U.* '48]

Tabulating the corresponding values of x and y from $y=5x^2+2x-1$, we have

x	0	1	−1	2	−2	...
y	−1	6	2	23	15	...

Let the axes XOX', YOY' intersect at right angles at the origin O. Draw the graph, which is a parabola, taking each small division on the graph paper as the unit.

Draw the graph of $y=3$, which is a st. line parallel to the x-axis at a distance of 3 units from it. The abscissæ of the two points where the graphs intersect are found to be $-1\cdot 7$ and $\cdot 7$ respectively.

Now, the equation $5x^2+2x-4=0$ may be written as $5x^2+2x-1=3$. Suppose $y=5x^2+2x-1$, then $y=3$.

Hence the above abscissæ of the points of intersection of the two graphs are the roots of $5x^2+2x-4=0$.

Ex. 13. Draw the graphs of $y=x^2$ and $x-y+6=0$. Hence solve $x^2-x-6=0$. [*D. B.* '14, '30, '40]

The graph of $y=x^2$ is a parabola. Draw the graph as in Ex. 1. The graph of $x-y+6=0$ is a st. line passing through the points $(0, 6)$, $(-2, 4)$, $(-6, 0)$, etc. Draw this graph. Let the graphs intersect at A and B.

Now the equation $x^2-x-6=0$ may be written as $x^2=x+6$, Let $y=x^2$ ...(1), then we hane $y=x+6$ or $x-y+6=0$ ...(2). $\therefore$ The parabola drawn is the graph of (1) and the st. line drawn is the graph of (2) Hence the abscissæ of A, B which are found to be -2 and 3, are the roots of $x^2-x-6=0$.

$\therefore$ The solution is $x=-2$ or 3.

[Ellipse]

Ex. 1. Draw the graph of $9x^2+16y^2=144$.

Here $9x^2+16y^2=144$ or $16y^2=144-9x^2=9(16-x^2)$,

Or, $y^2=\frac{9}{16}(16-x^2)$, $\therefore$ $y=\pm\frac{3}{4}\sqrt{16-x^2}$...(1)

Tabulating the corresponding values of x and y we have

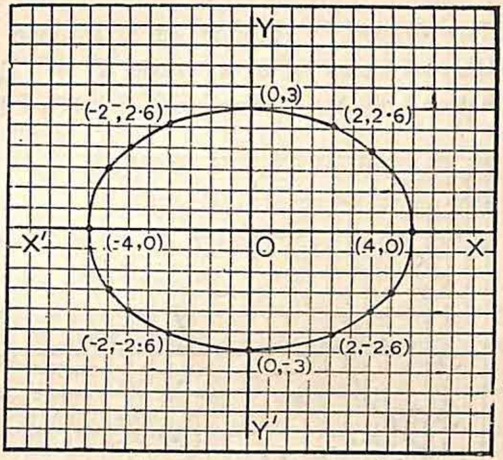
x	0	±4	±1	±2	±3	$\pm3{\cdot}5$	...
y	±3	0	$\pm2{\cdot}9$	$\pm2{\cdot}9$	$\pm2{\cdot}0$	$\pm1{\cdot}5$	...

The above pairs of points lie on the graph. Let the axes XOX', YOY' intersect at right angles at the origin O. Plot the points, taking two small divisions on the graph paper as the unit of length. Draw through these points a continuous freehand curve, which is the required graph. This curve is called an **Ellipse.**

[The students should observe the following points :—

(*a*) From (1) above, we find that the value of x cannot be greater than 4 or less than -4, for in that case y will be imaginary. Hence it is evident that the graph is limited between the two parallel lines $x=\pm4$. Again, for each value of x, y will have two values, equal in magnitude but opposite in sign. Hence the graph is symmetrical about the y-axis.

The given equation may also be written as $x=\pm\frac{4}{3}\sqrt{9-y^2}$. We find from it that the value of y cannot be greater than 3 and less than -3, for in that case x will be imaginary. Hence the graph lies between the parallel lines $y=\pm 3$. Again for each value of y, x will have two equal and opposite values. Hence the



[Fig. 21]

graph is also symmetrical about the *x-axis*. Thus the graph is a *closed curve*, being limited on all sides.

(*b*) Notice further that the given equation may be reduced to the form $\frac{x^2}{16}+\frac{y^2}{9}=1$. Again, the equation $4x^2+y^2-8x-4y=8$, may be written as $(4x^2-8x+4)+(y^2-4y+4)=16$, or, $4(x-1)^2+(y-2)^2=16$, and evidently its graph will also be an Ellipse. To draw this graph, transfer the origin to the point (1, 2), *i.e.*, taking the point (1, 2) as origin draw the graph of $4x^2+y^2=16$, and this graph will be the required graph.

(*c*) The graph of the function $\frac{3}{4}\sqrt{16-x^2}$ is the same as the graph of $y=\frac{3}{4}\sqrt{16-x^2}$.]

[Hyperbola]

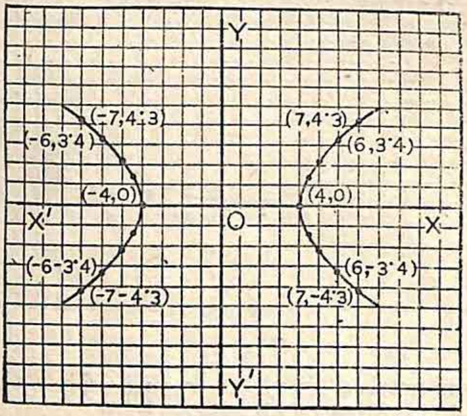
Ex. 1. Draw the graph of $9x^2-16y^2=144$.

Here $9x^2-16y^2=144$, or, $y^2=\frac{9x^2-144}{16}$, $\therefore\ y=\pm\frac{3}{4}\sqrt{x^2-16}$.

Tabulating the corresponding values of x, y we have

x	± 4	± 5	± 6	± 7	$\pm 4·5$	...
y	0	$\pm 2·25$	$\pm 3·4$	$\pm 4·3$	$\pm 1·5$	...

Let XOX', YOY' be the axes intersecting at right angles at the origin O. Let one side of a small square on the graph paper be the unit of length. Plot the above points. We have two sets of points. Now drawing a freehand curve through each set of points separately on either side of y-axis we get the graph of the given equation [fig. 22]. The graph is called a **Hyperbola**.



[Fig. 22]

[**N. B.** (1) For each value of x, y will have two equal values, one positive and the other negative. So the graph is symmetrical with respect to the x-axis. Similarly it is also symmetrical about the y-axis. (2) Here if $x=0$, y becomes imaginary. So the graph will never cut the *y-axis*. (3) If the value of x lies between $+4$ and -4, y becomes imaginary. Hence there is no part of the graph between the parallel lines $x=\pm 4$.

(4) The graph consists of two branches, extending to infinity on both sides of the y-axis.]

Ex. 2. Draw the graph of $9x^2-16y^2-18x-64y-91=0$.

$$9x^2-16y^2-18x-64y-91=0,$$
or, $$(9x^2-18x+9)-(16y^2+64y+64)-36=0,$$
or, $$9(x-1)^2-16(y+2)^2=36.$$

Transfer the origin to the pt. $(1,-2)$ and then draw the graph of $9x^2-16y^2=36$. This graph will be the graph of the given equation.

Ex. 3. Draw the graph of $xy=6$.

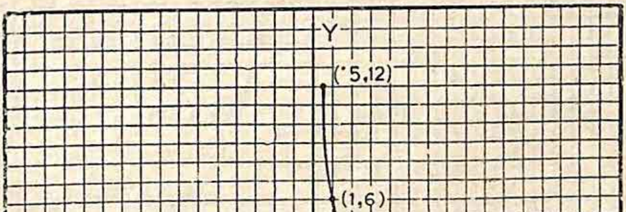
The given equation may be written as $y=\frac{6}{x}$.

Tabulating the values of x and y we have

x	·5	−·5	1	−1	2	−2	4	−4	6	−6	...
y	12	−12	6	−6	3	−3	1·5	−1·5	1	−1	...

The above points are on the graph.

Let XOX', YOY' be the axes intersecting at right angles at the origin O. Plot the points, taking one small division on the graph paper as the unit of length. We have two sets of points, one lying in the first quadrant and the other in the third. Drawing a freehand curve through each set of points separately, we get the required graph, which is called a **retangular hyperbola.**

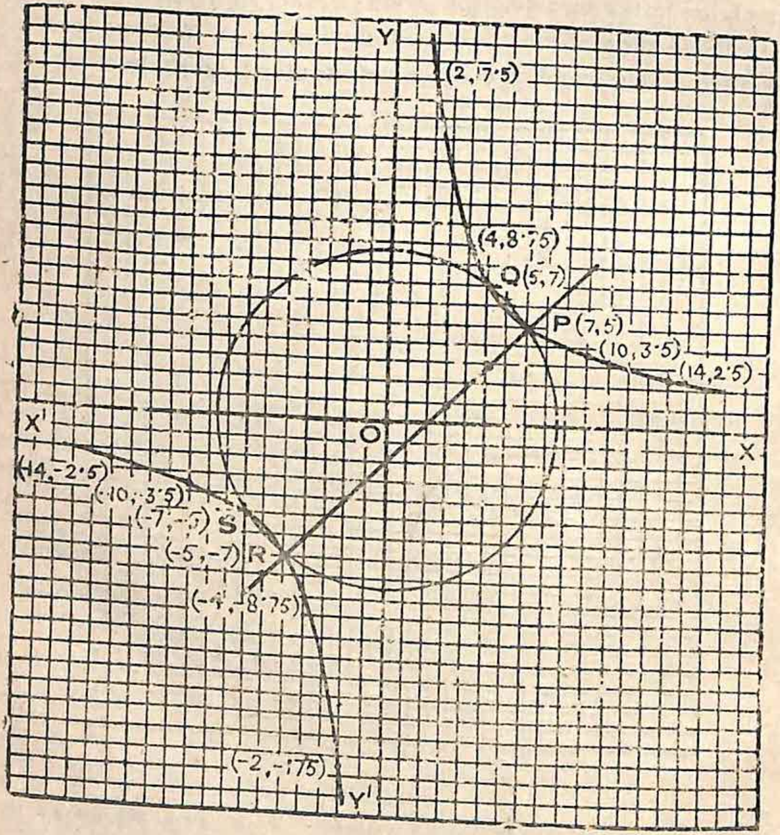


[Fig. 23]

[**N. B.** Note the following points : (1) Here when $x=0$, $y=\infty$ and when $x=\infty$, $y=0$. If x is positive, y is also positive and when x is negative, y is negative. So the graph has two branches, one lying in the first quadrant and the other in the third. (2) As the values of x increase in the positive direction, the values of y become smaller and are positive. Ultimately y becomes zero, when $x=\infty$. Hence the graph continually approaches the *x-axis* but does not actually touch it until x is ∞. (3) As x becomes smaller and smaller, the graph approaches more and more to the positive end of the *y-axis*.]

Ex. 4. Solve the following pairs of equations graphically :

$$\text{(i)}\quad \left.\begin{aligned} x^2+y^2&=74\\ xy&=35\end{aligned}\right\} \quad\text{and}\quad \text{(ii)}\quad \left.\begin{aligned} x-y&=\ 2\\ xy&=35\end{aligned}\right\}$$



[Fig. 24]

Here the graphs of the following three equations are to be drawn :

(1) $x^2+y^2=74$ (it is a circle)

(2) $xy=35$ (it is a rectangular hyperbola)

(3) $x-y=2$ (it is a st. line).

From (1) we have $y^2=74-x^2$, or, $y=\pm\sqrt{74-x^2}$.

$\because$ $7^2+5^2=74$, $\therefore$ $x=7$, $y=5$ satisfy the equation.

On the graph paper plot a pt. P whose co-ordinates are (7, 5) and draw a circle with centre O (origin) and with radius OP. This circle is the graph of equation-(1).

(2) $\because$ $xy=35$, $\therefore$ $y=\dfrac{35}{x}$.

Tabulating the values of x, y we have

x	5	7	−5	−7	4	−4	10	−10	...
y	7	5	−7	−5	8·75	−8·75	3·5	−3·5	...

Plot the points and draw the rectangular hyperbola.

(3) From $x-y=2$, or, $y=x-2$, we have

x	5	3	−2
y	3	1	−4

Draw a st. line through those points. It is the graph of (3).

The three graphs are drawn, taking one small division on the graph paper as the unit of length.

We find that the circle cuts the rectangular hyperbola at P, Q, R and S, whose co-ordinates are (7, 5), (5, 7), (−5, −7) and (−7, −5) respectively.

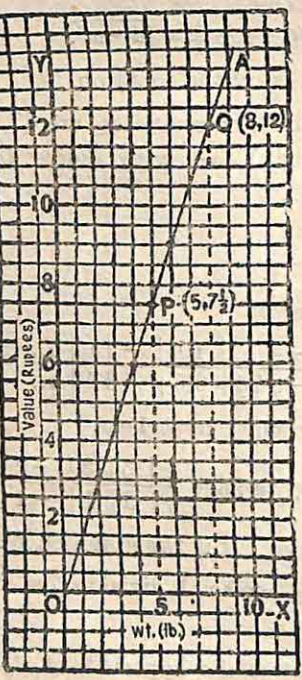
Hence the solution of the first pair of equations is

$$\left.\begin{matrix}x=7\\y=5\end{matrix}\right\},\quad \left.\begin{matrix}x=5\\y=7\end{matrix}\right\},\quad \left.\begin{matrix}x=-5\\y=-7\end{matrix}\right\}\text{ and }\left.\begin{matrix}x=-7\\y=-5\end{matrix}\right\}.$$

Again, the st. line cuts the rectangular hyperbola at two points, whose co-ordinates are (7, 5) and (−5, −7). Hence the solution of the second pair of equations is $x=7$ and $y=5$, or, $x=-5$ and $y=-7$.

Graphical Solution of Problems

Ex. 1. If 1 ℔ of tea costs Re. 1. 8 as., find by means of a graph (*i*) the price of 5 ℔s. of tea and (*ii*) the quantity of tea that can be had for Rs. 12.



Let y rupees be the price of x pounds of tea.

Here we have the cost of 1 ℔. of tea is Rs. $\frac{3}{2}$.

$\therefore$ The cost of x pounds of tea is $\dfrac{3x}{2}$ rupees.

$\therefore$ We have $y=\dfrac{3x}{2}$, which is the equation of the required graph.

From $y=\dfrac{3x}{2}$ we have

x	2	4	6	...
y	3	6	9	...

Let one small division on the graph paper along the x-axis represent 1 ℔. and two small divisons along the y-axis represent one rupee.

The st. line OA is drawn through the above points. It is the graph of $y=\dfrac{3x}{2}$.

(*i*) The abscissa and the ordinate of any point on this graph will give the weight of tea (in pounds) and its price (in rupees).

Now, we find from the graph that the ordinate of the point (P), whose abscissa is 5 units, is $7\frac{1}{2}$ units.

$\therefore$ The required cost of 5 ℔s. of tea $=$ Rs. $7\frac{1}{2}$.

(*ii*) Again, we find that the abscissa of the point (Q), whose ordinate is 12 units, is 8 units.

∴ 8 lbs. of tea can be had for Rs. 12.

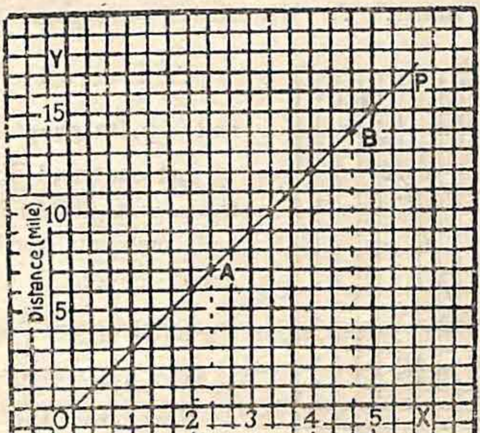
[**N. B.** The st. line OA is called the **price-graph** of tea.]

Ex. 2. A man walks at the rate of 3 miles an hour. Draw his motion-graph and find from it (*i*) how far he will walk in 2 hrs. 20 mins. and (*ii*) in what time he will walk 14 miles.

Suppose the man walks y miles in x hours. Here he goes 3 miles in 1 hr., ∴ he goes $3x$ miles in x hours.

∴ we have y miles $=3x$ miles.

∴ the graph of the equation $y=3x$ is the required **motion-graph.**



Let three small divisions on the graph paper along the x-axis denote 1 hour (*i.e.*, 1 division represents 20 minutes), and one small division along the y-axis denote 1 mile.

Draw, OP the graph of $y=3x$. Then OP is the required graph. The abscissa and the ordinate of any point on this graph will respectively denote the time and the distance.

(*i*) 2 hrs. 20 mins. $= \frac{7}{3}$ hrs. From the graph we find that the ordinate of the point (A), whose abscissa is 7 small divisions (*i.e.*, $\frac{7}{3}$ units), is 7 divisions or 7 units.

∴ The man walks 7 miles in 2 hrs. 20 mins.

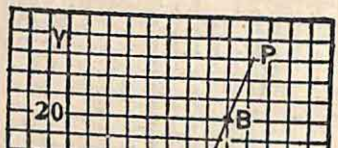
(*ii*) Again, we find that the abscissa of the point (B), whose ordinate is 14 units, is 14 divisions or $\frac{14}{3}$ units.

∴ The man takes $\frac{14}{3}$ hrs. or 4 hrs. 40 mins. to walk 14 miles.

[**N. B.** As the man walks at a uniform rate, the motion graph is a st. line.]

Ex. 3. If the price of two mangoes be 5 annas, find graphically the price of 5 mangoes and the number of mangoes that can be had for Re. 1.4 as.

Let the price of x mangoes be y annas. Here the price of 2 mangoes is 5 annas. ∴ the price of x mangoes $= \frac{5x}{2}$ annas.



∴ The graph of $y = \frac{5x}{2}$ will be the **price-graph** here.

Let one small division along the x-axis denote 1 mango and one such division along the y-axis denote 1 anna. Draw the graph of $y = \frac{5x}{2}$ and let OP be the graph.

Now, from the graph it is found that

(*i*) The ordinate of the point (A), whose abscissa is 5 units, is $12\frac{1}{2}$ units.

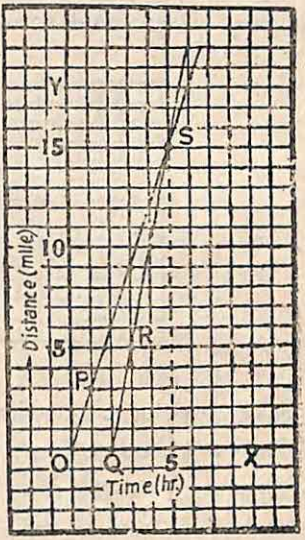
∴ The price of 5 mangoes $= 12\frac{1}{2}$ annas.

(*iii*) Again, Re. 1. 4 as. $= 20$ annas. The abscissa of the point (B), having 20 units as ordinate, is 8 units.

∴ 8 mangoes can be had for Re. 1. 4 as.

Ex. 4. *A* starts walking at 8 A. M. at the rate of 3 miles an hour. After 2 hours *B* runs after him at the rate of 5 miles an hour. Find graphically when and where *B* will overtake *A*.

Let one small division along the x-axis denote 1 hr. and that along the y-axis denote 1 mile.



Suppose *A* started from the origin *O* and walked uniformly at the rate of 3 miles per hour. Let *P* be a point whose co-ordinates are (1, 3). Its abscissa 1 unit denotes 1 hr. and the ordinate 3 units denotes 3 miles. Hence the pt. *P* must lie on the motion graph of *A*.

∴ The st. line *OP* produced represents the motion graph of *A*.

Again, *B* started 2 hours after *A*. Though 2 hrs. elapsed, *B* did not move any distance in these 2 hrs., *i.e.*, his motion in these 2 hrs. is 0 mile.

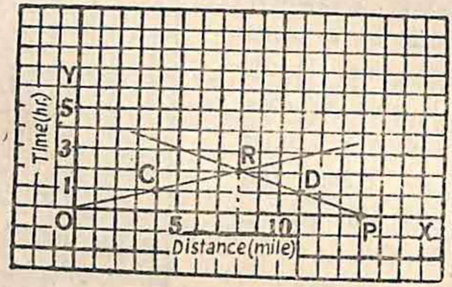
Let *Q* be a pt. whose co-ordinates are (2, 0). Then the st. line *OQ* represents the motion-graph of *B* for those 2 hrs. (8 A. M. to 10 A. M.). After this *B* walked uniformly 5 miles per hour. Now taking *Q* as the origin, plot a pt. *R* whose co-ordinates are (1, 5). Then the st. line *QR* produced represents the motion graph of *B* after the first 2 hours.

The st. lines *OP*, *QR* intersect at *S*. The co-ordinates of *S*, with *Q* as origin, are (5, 15). Here the abscissa 5 denotes 5 hrs. and the ordinate 15 denotes 15 miles.

∴ *B* overtook *A* at a distance of 15 miles from the starting place 5 hrs. after *A* had started, *i.e.*, at 1 P. M.

Ex. 5. A and B are at a distance of 14 miles and start at the same time cycling towards each other at the rates of 4 and 3 miles per hour respectively. Find by a graph when and where they will meet.

On the graph paper draw the x-axis and y-axis with O as origin. Let 1 small division along the x-axis denote 1 mile and that along the y-axis denote 1 hour.



Take a point P on the x-axis 14 units away from O. Then OP denotes 14 miles.

Suppose A walks from O towards P at the rate of 4 miles per hr. and B from P towards O at 3 miles per hr. The motion graphs of A and B will be the straight lines drawn through O and P respectively.

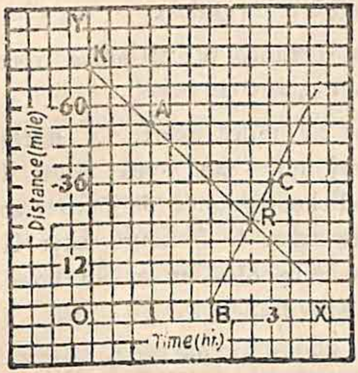
Now, $\because$ A goes 4 miles in 1 hr., $\therefore$ his motion-graph will pass through the pt. C (4, 1). Hence the st. line OC produced is the motion-graph of A.

Again, in the case of B's motion, the direction towards O is positive. Plot a pt. D whose co-ordinates are (3, 1). Then the st. line PD produced is the motion-graph of B. The two graphs intersect at R, whose co-ordinates are (8, 2) representing 8 miles and 2 hours respectively.

Hence, they will meet after 2 hours at a distance of 8 miles from the starting place of A.

Ex. 6. A mail train starts from Howrah at 9 P. M. and reaches Kharagpur, a distance of 72 miles, at 11 P. M. A passenger train starts from Kharagpur at 7 P. M. and reaches Howrah at 11 P.M. Find graphically when and where they meet. [*W. B. S. F.* 1955 (Addl.)]

Let M be the mail train and P the passenger train. M runs $(72 \div 2)$ or 36 miles in 1 hr. and P runs $(72 \div 4)$ or 18 miles in 1 hr.



On the graph paper let OX and OY be the axes. Let 3 small divisions along the x-axis denote 1 hour and 1 small division along the y-axis denote 6 miles.

Suppose the position of Howrah is at O, the origin, and the position of Kharagpur is at K which is a pt. on the y-axis at a distance of 12 divisions (*i. e.*, 72 miles) from O.

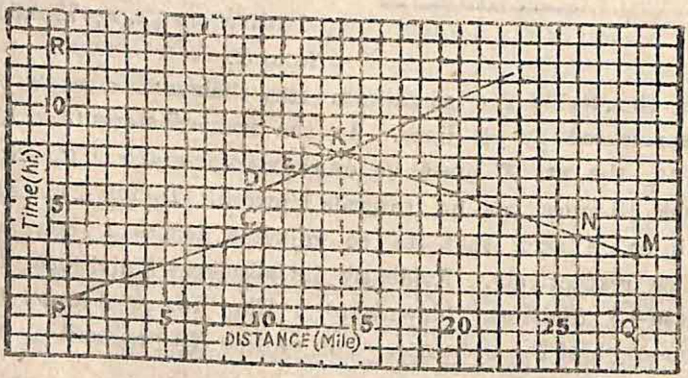
As the trains run at uniform rates, their motion-graphs will be represented by st. lines. As the train P starts from K at 7. P. M., the pt. K must lie on the motion-graph of P. Now, P runs 18 miles per hr. towards Howrah. Taking K as origin plot a pt. A such that its co-ordinates denote 1 hour and 18 miles respectively. Evidently its abscissa $= 3$ divisions, and ordinate $= 3$ divisions (downwards). The pt. A lies on the motion-graph of P. Join KA. The st. line KA produced is the motion-graph of P.

Again, as the train M starts at 9 P. M., it runs 0 mile during the first 2 hours (from 7 P. M. to 9 P. M.). Plot a pt. B on OX 6 small divisions away from O. Then OB represents 2 hours. The pt. B represents the starting place of M and lies on its motion-graph. M runs 36 miles in 1 hr. Taking B as origin, plot a pt. C such that its co-ordinates denote 1 hour and 36 miles respectively. Evidently, its abscissa $=3$ divisions and the ordinate $=6$ divisions. The pt. C is then on the motion-graph of M. Hence the st. line BC produced represents the motion graph of M.

The two graphs intersect at R. With respect to the origin O, the abscissa of $R=8$ divisions (*i.e.*, $\frac{8}{3}$ hrs. or 2 hrs. 40 mins.) and the ordinate $=4$ divisions (*i.e.*, 4×6 or 24 miles).

Hence, the two trains meet at a distance of 24 miles from Howrah 2 hrs. 40 mins. after 7 P.M., *i.e.*, at 9-40 P.M.

Ex. 7. P and Q are two places 29 miles apart. A starts from P and walks towards Q at $2\frac{1}{2}$ miles an hour. After 4 hours he takes rest for 2 hours and then resumes his journey at the rate of 2 miles an hour. B starts from Q 3 hours after A leaves P and walks towards P at the rate of 3 miles an hour. Find graphically when and where they will meet.



Let the straight lines PQ and PR, perpendicular to each other, represent the x-axis and y-axis respectively on the graph paper. Let one small division along PQ denote 1 mile and that along PR denote 1 hour. Here $PQ=29$ small divisions denoting 29 miles.

(i) Starting from P, A has gone $(2\frac{1}{2}\times 4)$ or 10 miles in 4 hrs. Plot a point C (10, 4) so that its abscissa 10 denotes 10 miles and the ordinate 4 denotes 4 hrs. Join PC.

PC is the motion graph of A for the first 4 hours.

After this A rests for 2 hours while he covers no distance. So the motion-graph for these 2 hrs. will be such that the abscissa of any point on it will indicate 10 miles. Now plot a pt. D (0, 2) taking C as origin. Thus CD is the motion-graph of A for 2 hours of rest.

After this A moves towards Q at 2 miles an hour. Plot a point E (2, 1) taking D as origin. So the st. line DE produced represents the motion-graph of A after rest.

(ii) Again, B starts from Q 3 hrs. after A and it has covered no distance during these 3 hrs. Plot a pt. M with Q as origin so that its abscissa denotes 0 mile and ordinate 3 hours. The st. line QM is the motion-graph of B for his 3 hours of rest.

Since then B moves from M towards P at 3 miles per hour. Taking M as the origin plot a pt. N such that its abscissa and ordinate indicate 3 miles and 1 hour respectively. So the st. line MN produced is the motion-graph of B after rest.

DE and MN intersect at K. The co-ordinates of K with respect to P as origin denotes 14 miles and 8 hours respectively.

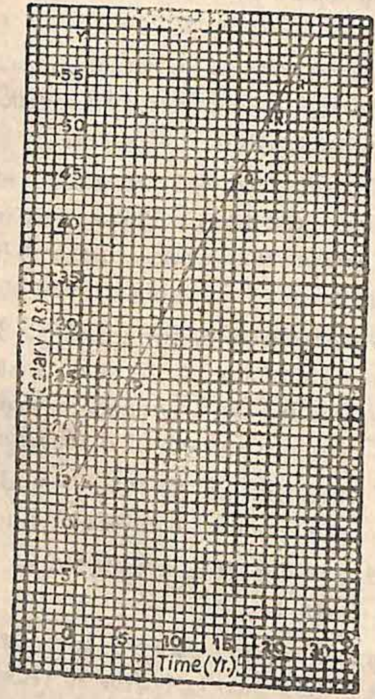
Hence A and B will meet at a distance of 14 miles from P 8 hours after A starts.

Ex. 8. A man's salary after 5 years of service is 25 rupees, after 15 years it is 45 rupees and after 20 years it is 55 rupees. If his salary was increasing at a uniform rate, show by graph his starting salary and the salary he will get after 18 years of service. [*Pat. U.* 1945]

As the salary increases at a uniform rate, the graph will be a st. line, and the points, denoting 5 years and Rs. 25, 15 yrs. and Rs. 45, 20 yrs. and Rs. 55 will lie on the graph.

Let one small division along the x-axis denote 1 year and that along the y-axis denote one rupee. Now, plot a pt. (5, 25), so that its abscissa denotes 5 years and ordinate denotes Rs. 25. Similarly plot the points Q and R whose co-ordinates are (15, 45) and (20, 50) respectively. Join these points.

Let RQP produced meet the y-axis (OY) at A. Then the st. line AR is the required graph.



The abscissa of A is 0 and its ordinate is 15. Hence the initial salary was Rs. 15.

Again to determine the salary he will get after 18 years' service, find from the graph the ordinate of a point whose abscissa is 18 units. From the graph we find that the abscissa of K is 18 and its ordinate is 51. Hence the required salary after 18 years of service is Rs. 51.

Ex. 9. The population of a certain town is given by the following table :—

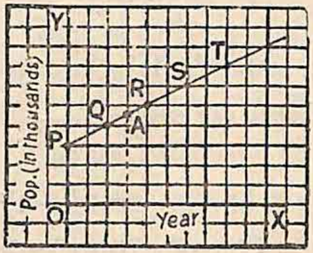
Year	...	1905	1915	1925	1935	1945
Population in thousands	...	15	20	25	30	35

Draw a graph and read the population of the town in 1920.
[*Pat. U.* '46]

As the population increases uniformaly, its graph will be a st. line.

Let one small division along the x-axis denote 5 years and that along the y-axis denote 5 thousand (population).

Let 1905 be taken as the initial year. Plot a point $P\ (0, 5)$ so that its abscissa 0 indicates the initial year 1905 and the ordinate 3 denotes the population (3×5) or 15 thousand.



Now the co-ordinates of the point that will represent the population 20 thousand in 1915 must be (2, 4), for the abscissa (2 units) will denote 10 yrs. (after 1905) and the ordinate (4 units) will denote 20 thousand. Let Q be the point.

Similarly plot the points $R(4, 5)$, $S(6, 6)$, $T(8, 7)$ to represent the population in 1925, 1935, 1945 respectively. Evidently the st. line PT is the required graph.

To determine the population in 1920 from the graph, find the ordinate of the point whose abscissa in 3 units $(1920 - 1905 = 15$ yrs. $= 3$ units). It is found from the graph that the co-ordinates of A are $(3, 4\frac{1}{2})$. Its ordinates ($4\frac{1}{2}$ units) denotes $(5 \times 4\frac{1}{2})$ thousand or 22500 people.

$\therefore$ The required population in $1920 = 22500$.

Exercise 12

Draw the graphs of :—

1. (i) $x^2 - y^2 + 6y - 9 = 0$

(ii) $2y^2 - 5y + 2 = 0$

(iii) $xy = 0$

(iv) $x^2 - xy - 6y^2 - 11x + 18y + 24 = 0.$

Solve graphically :—

2 (*a*) $4x + 3y = 15$ and $x - y = 2$

(*b*) $2x + 3y = 13$ and $3x - 2y = 13$ [*P. U.* '24]

(*c*) $\frac{3x-4}{2} = 3x - \frac{1}{2}$ [*P. U.* '25]

(*d*) $\frac{2x+4}{2} = 2x - 1$ (*e*) $x - 2 = 5$

(*f*) $x^2 - 4y^2 - x + 8y - 3 = 0.$

Draw the graphs of :—

3. (i) $x^2 + y^2 = 36$ [*C. U.* '41]

(ii) $(x-2)^2 + (y-1)^2 = 25$

(iii) $x^2 - 8y + y^2 - 6x - 24 = 0$ [*D. B.* '26]

4. $(x+1)^2 + (y-3)^2 = 16$

5. $x^2 + y^2 = 48$ 6. $x^2 + y^2 = 40$

7. $y^2 = 4x$ [*C. U.* '25, '37, '40, '42]

8. $y = (x+1)^2$ [*C. U.* '27]

9. $y = x^2 - x - 6$ 10. $y = 4x^2$

11. $y^2 + 4x = 0$ [*C. U.* '28, '39]

12. $5x^2 - 10x + 9$ [*C. U.* '47]

13. $x^2 + y^2 - 2x - 4y = 20$

14. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ [*C. U.* '44]

15. $4x^2+9y^2=64$

16. $x^2-y^2=1$ **17.** $x^2-y^2=9$

18. $xy=12$ [*A. U.* '16] **19.** $x^2=25$

20. $3x^2-5xy+2y^2=0.$

21. Draw the graphs of $x^2+y^2=169$ and $x+y=17$, and find the co-ordinates of their points of intersection. [*G. U.* '57]

22. Draw the graphs of $y=x^2$ and $y=2x-1$, and determine where they meet. [*C. U.* '45]

23. Trace the graphs of $y=x$ and $y=\frac{x^2}{4}$ and derermine the points where they intersect. [*E. B. S. B.* '51]

24. Trace the changes in sign and magnitude of x^2-4x+3 as x increases from $-\infty$ to $+\infty$.

25. Draw the graphs of $4y=x^2$ and $2y=x+4$ between the values $x=-4$ and $x=4$ and find from the graphs their points of intersection. [*C. U.* '51]

26. Draw the graph of $y=4x^2-3x+2$ and use it to find the minimum value of y. [*E. B. S. B.* '49]

27. Draw the graphs of $x^2+y^2=25$ and $x+y=7$; and measure the co-ordinates of their points of intersection. [*C. U.* '12 ; *D. B.* '25]

28. Draw the graph of $2+x-2x^2$ and find from it the maximum value of the function. [*D. B.* '40]

29. Draw the graphs of $y^2=8x$ and $y=2x+1$ and indicate their common point, if any. [*C. U.* '50]

30. Trace the graph of $y=x^2-6x+10$ and find the least value of y. [*D. B.* '24]

31. Solve graphically $x^2-x-2=0$. [*C. U.* '49 ; *G. U.* '48]

32. Solve graphically $2x^2-5x+3=0$. [*E. B. S. B.* '48]

33. Prove graphically that x^2-4x+5 is positive for all real values of x.

34. Draw the graphs of $y=x^2$ and $y=2x+3$. Hence solve graphically $x^2=2x+3$. [*G. U.* '49]

35. Draw the graphs of $4y=x^2$ and $2y=x+4$ between $x=-4$ and $x=+4$, and measure the length of the chord of intersection. [*G. U.* '14]

36. Draw the graphs of $y=x^2+3$ and $y=4x$ and hence find the solution of $x^2-4x+3=0$. [*D. B.* '37]

[Hints : $x^2-4x+3=0$, or, $x^2+3=4x$; draw the graphs of $y=x^2+3$ and $y=4x$. The abscissæ of the points of their intersection are the required roots.]

37. In the same diagram draw the graphs of $2x+1$ and x^2. From your graphs read as accurately as you can the value or values of x which will make $x^2=2x+1$. [*C. U.* '30]

38. Draw the graphs of $x^2+y^2=16$ and $x+y=2$ and measure the length of the chord of intersection. [*C. U.* '13]

39. Trace the graph of $y=x^2-4x+5$ from $x=0$ to $x=4$ and find the least value of y. [*C. U.* '18]

40. Draw the graph of $2x^2-7x-3$ between $x=-1$ and $x=+5$. From your graph determine the roots of $2x^2-7x+3=0$.

41. Draw the graphs of $y=x^2+3x$ and $y=-2$. Hence find the roots of $x^2+3x+2=0$. [*C. U.* '46]

[Hints : $x^2+3x+2=0$ or $x^2+3x=-2$. Let $y=x^2+3x$ and $y=-2$. Then the abscissæ of the points of intersection of the graphs of the two equations are the required roots.]

42. Draw the graph of $y^2=4x$ and prove that their is no part of the graph on the negative side of the axis of x. [*C. U.* '25]

43. Solve the equation $x^2-6x+8=0$ by drawing the graph of $y=x^2$. [*C. U.* '47]

[Hints : $x^2-6x+8=0$, or, $x^2=6x-8$. Let $y=x^2$, $\therefore$ $y=6x-8$. Now draw the graphs of $y=x^2$ and $y=6x-8$.]

44. Draw the graphs of $y=x^2$ and $x=y^2$ and find the co-ordinates of their points of intersection. [*C. U.* '32]

45. Draw the graph of x^2+2x+5 from $x=-4$ to $x=2$. Find from the graph the minimum value of the function and the value of x that gives the minimum value. [*D. B.* '32]

46. Draw the graph of x^2+2x+3 from $x=-5$ to $x=3$. Read off from the graph answers to the following questions :

(i) What is the least value of the function ?

(ii) What value of x gives this least value ?

(iii) For what value of x does the function take the value 4 ? [*M. U.* '26]

47. Plot the graph of $xy=80$ between the values $x=\pm 20$. [*M. U.* '13]

Draw the graphs of :—

48. $x=\frac{1}{2}x^2-3$. [*M. U.* '12] 49. $\sqrt{x}$

50. $4x^2-16x+y^2+2y-8=0$.

51. $9x^2-18x-16y^2-32y-151=0$.

52. Find the co-ordinates of the point where the graph of $3x-4y+25=0$ touches the graph of $x^2+y^2=25$.

53. Solve graphically the equation $2x^2-9x-5=0$. [*C. U.* '50]

54. A cyclist starts at 8 A. M. on a ride of 20 miles at 5 miles an hour. Draw a graph showing the relation between the distance travelled and the time taken to cover that distance.

55. If two oranges cost 3 annas, find graphically (1) the cost of 7 oranges and (2) how many oranges can be had for Re. 1. 5 as.

56. A man starts walking at 10 A. M. at the rate of 5 miles an hour. After 2 hours his son cycles after him at the rate

of 7 miles an hour. Find graphically when and where the son will overtake the father. [*A. U.* '43]

57. A train P starts from Howrah and runs at 30 miles an hour. Another train Q starts 20 minutes after P and runs on a parallel line at 50 miles an hour. (1) When and where will Q overtake P ? How far will they be apart 12 minutes after Q starts ?

58. A walks at the rate of 4 miles an hour and takes rest for 18 minutes at the end of every hour. Two hours later B runs in the same direction at the rate of 6 miles an hour. Find graphically when and where they will meet. [*N. U.* '47]

59. A train starts from Howrah for Magra (30 miles distant) at 8-30 A. M. and travels 40 miles an hour. Another train starts from Magra for Howrah at 8-45 A. M. and travels at 10 miles an hour. When and where do they meet ?

60. Two friends X and Y leave their places A and B respectively to meet each other. X starts at 9 A. M. and travels at 12 miles per hour, while Y starts at 11 A. M. and travels at 30 m. p. h. They meet at 12-20 P. M. Draw graphs of their travels and read from them the distance between A and B. [*N. U.* '48]

61. Two pipes can fill a cistern in 4 and 5 hours respectively. How long will they take to fill it if they are opened together ?

62. At noon A starts to cycle from P to Q, a distance of 40 miles. He rides 6 miles an hour, resting for an hour after riding 12 miles. At 3 P. M. B starts from P at 10 miles an hour. Find graphically (*a*) when and where B overtakes A and (*b*) their distance apart at 5 P. M. [*A. U.* '45]

63. The salary of an officer increases each year by a fixed sum. After 5 years of service his salary is raised to Rs. 120 and after 12 years to Rs. 176. Draw a graph from which his salary may be read off for any year and determine from it

(*i*) his initial salary and (*ii*) the salary he should receive for his 21st year. [*D. B.* '45]

64. The population of a town is given by the following table :—

Year	1920	1930	1940	1950
Population in thousands	12	17	22	27

Find graphically the population of the town in 1945.

65. Given that 5 kilograms are approximately equal to 11 pounds, show graphically how to express any number of kilograms in pounds. Express from the graph 7·5 pounds in kilograms and 7 kilograms in pounds.

66. Draw a graph representing the squares of the numbers from 0 to 5.

Solve graphically :—

67. $x^2+y^2=41,\ y-2x=-3.$

68. $\frac{x^2}{4}+x-8=0.$

69. $\left.\begin{aligned} x-2y&=8\\ xy&=24\end{aligned}\right\}$ 70. $\left.\begin{aligned} x^2+y^2&=73\\ xy&=24\end{aligned}\right\}$

71. Draw the graphs of (i) $\frac{x^2}{9}+\frac{y^2}{16}=1$, plotting at least 8 points, and (ii) $x+y=5$, plotting at least 3 points, with the same axes of co-ordinates and same scale. Show from your graph that (ii) touches (i). Write down the co-ordinates of the point of contact. [*W.B.S.F.* '59]

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
★ CALCUTTA-27 ★

GEOMETRY

57. Projections : The projection of a *point* on a given straight line is the foot of the perpendicular on it from the given point.

In fig.-A, Aa is drawn perpendicular from the point A on the straight line PQ. Here the point a is the projection of the point A on PQ. In fig.-B, the projection of the point A on the straight line PQ is the point A itself, as A lies on PQ.

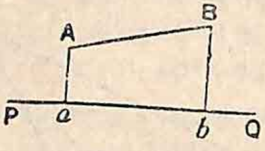


Fig. A

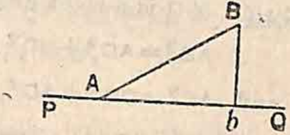


Fig. B

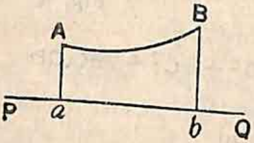


Fig. C

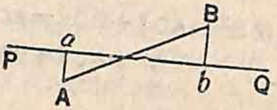


Fig. D

The projection of a *line* AB on the straight line PQ is that portion of PQ which is between the feet of the perpendiculars drawn from A and B to PQ.

Suppose perpendiculars Aa and Bb are drawn from A and B respectively on PQ. Then in figs. A and C, ab is the projection of AB on PQ. In fig.-B, Ab is the projection of AB on PQ. In fig.-D, suppose AB cuts PQ in O, then Oa and Ob respectively are the projections of AO and BO on PQ.

These projections are called *Orthogonal Projections*.

Theorem 1

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of those sides and the projection of the other side upon it.

Let ABC be a triangle in which $\angle ACB$ is obtuse.

Let AD be the perpendicular from A to BC produced, so that CD is the projection of the side AC on BC.

It is required to prove that $AB^2 = AC^2 + BC^2 + 2BC.CD$.

Proof: $\because$ $\angle D$ is a right angle,

$\therefore$ $AB^2 = AD^2 + BD^2$

and $AC^2 = CD^2 + AD^2$.

Again, $\because$ BD is divided internally at C(*i.e.*, $\because BD = BC + CD$),

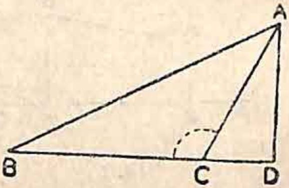


Fig. 1

$$\therefore \quad BD^2 = BC^2 + CD^2 + 2BC.CD.$$

$$\therefore \quad AB^2 = AD^2 + BD^2 = AD^2 + CD^2 + BC^2 + 2BC.CD$$
$$= AC^2 + BC^2 + 2BC.CD.$$

Theorem 2

In any triangle, the sqaure on the side opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.

Let ABC be any triangle, having angle C an acute angle. Then AB is the side opposite to $\angle C$, and BC and AC are the other two sides.

Let AD be the perpendicular from A to CB (in fig. i) or CB produced (in fig. (ii), so that CD is the projection of AC on BC.

To prove that, $AB^2 = AC^2 + BC^2 - 2BC.CD$.

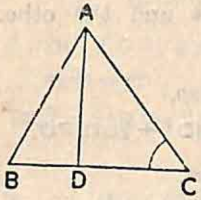


Fig. 2(i)

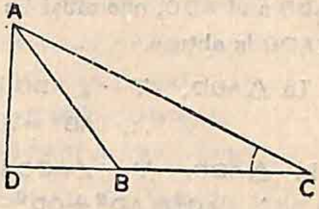


Fig. 2(ii)

Proof : $\because$ $\angle D$ is a right angle, $\therefore$ $AB^2 = AD^2 + BD^2$ and $AC^2 = AD^2 + CD^2$.

Again, $BD = BC - CD$, or, $BD = CD - BC$,

$$\therefore \quad BD^2 = BC^2 + CD^2 - 2BC.CD.$$

$$\therefore \quad AB^2 = AD^2 + BD^2 = AD^2 + CD^2 + BC^2 - 2BC.CD$$
$$= AC^2 + BC^2 - 2BC.CD.$$

[**N. B.** If in a triangle ABC,

$\angle C$ is a right angle, then $AB^2 = AC^2 + BC^2$,
$\angle C$ is obtuse, " $AB^2 > AC^2 + BC^2$,
$\angle C$ is acute, " $AB^2 < AC^2 + BC^2$.]

Apollonius' Theorem

In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

Let ABC be a triangle in which the median AD bisects BC.

To prove that, $AB^2 + AC^2 = 2BD^2 + 2AD^2$.

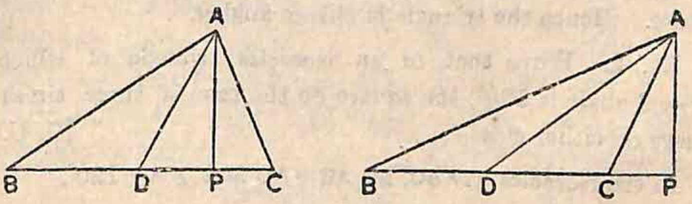


Fig. 3

Construction : From A draw AP perpendicular to BC or BC produced, so that DP is the projection of AD on BC.

Proof : Suppose AB and AC to be unequal. Then of the angles ADB and ADC, one must be obtuse and the other acute. Here $\angle ADB$ is obtuse.

In $\triangle ABD$, $\because$ $\angle ADB$ is obtuse,

$$\therefore \quad AB^2 = AD^2 + BD^2 + 2BD.PD.$$

Again, in $\triangle ADC$, $\because$ $\angle ADC$ is acute,

$$\therefore \quad AC^2 = AD^2 + CD^2 - 2CD.PD$$

$$= AD^2 + BD^2 - 2BD.PD \ (\because CD = BD)$$

$$\therefore \quad AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

If $AB = AC$, then the median AD is perpendicular to BC.

Then $AB^2 + AC^2 = (AD^2 + BD^2) + (AD^2 + DC^2)$

$$= AD^2 + BD^2 + AD^2 + BD^2$$

$$= 2AD^2 + 2BD^2.$$

Miscellaneous Solutions (1)

Ex. 1. Prove that a triangle whose sides are 2, 3 and 4 inches is an obtuse-angled triangle. [C. U. '33]

The obtuse angle being the greatest angle in a triangle, the side opposite to it is the greatest side of the triangle. If the square on the greatest side is greater than the sum of the squares on the other two sides, the angle opposite to it is obtuse.

Here, $4^2 = 16$ and $3^2 + 2^2 = 13$, $\therefore$ $4^2 > (3^2 + 2^2)$.

$\therefore$ The angle opposite to the side of 4 inches must be obtuse. Hence the triangle is obtuse-angled.

Ex. 2. Prove that in an isosceles triangle of which the vertical angle is 120°, the square on the base is three times the square on either side. [C. U. '16]

In the isosceles $\triangle ABC$, let $AB = AC$ and $\angle A = 120^\circ$.

To prove that $BC^2 = 3AB^2$.

Draw CD perpendicular from C to BA produced, so that AD is the projection of AC on BA.

Proof : $\because$ $\angle BAC$ is obtuse,

$$\therefore \quad BC^2 = AB^2 + AC^2 + 2AB.AD = 2AB^2 + 2AB.AD.$$

Now in $\triangle ADC$, $\angle D$ is a right angle and $\angle CAD = 180^\circ - 120^\circ = 60^\circ$,

$\therefore$ $\angle ACD = 30^\circ$, $\therefore$ $AD = \frac{1}{2}AC = \frac{1}{2}AB$.

$$\therefore \quad BC^2 = 2AB^2 + 2AB.AD = 2AB^2 + 2AB.\tfrac{1}{2}AB$$
$$= 2AB^2 + AB^2 = 3AB^2.$$

Ex. 3. If DE is drawn parallel to the base BC of an isosceles $\triangle ABC$, prove that the difference of the squares on BE and CE is equal to the rectangle contained by BC and DE. [C. U. '38]

In $\triangle ABC$, $AB = AC$ and $DE \parallel BC$.

To prove that $BE^2 - CE^2 = BC.DE$.

From D and E draw DP and ER perpendicular to BC.

Proof : Here $DERP$ is a parallelogram,

$\therefore$ $DP = ER$ and $DE = PR$,

In $\triangle DBP$ and $\triangle ERC$, $\angle P = \angle R$, $\angle B = \angle C$ and $DP = ER$,

$\therefore$ $PB = RC$. Now, since $\angle C$ of $\triangle BEC$ is acute,

$$\therefore \quad BE^2 = BC^2 + CE^2 - 2BC.CR.$$
$$\therefore \quad BE^2 - CE^2 = BC^2 - 2BC.CR = BC(BC - 2CR)$$
$$= BC(BC - CR - BP) = BC.PR = BC.DE.$$

Ex. 4. In a $\triangle ABC$, AD is perpendicular drawn to the base BC and O is the middle point of BC. Prove that the difference $AB^2 \sim AC^2 = 2BC.OD$. [C. U. '30, '46, '51 ; D. B. '41]

[**Hints** :— $\angle AOB > \angle D$, $\therefore$ $\angle AOB$ is obtuse and $\angle AOC$ is acute.

$\therefore$ $AB^2 = AO^2 + BO^2 + 2BO.OD \cdots (1)$

In $\triangle AOC$, $AC^2 = AO^2 + CO^2 - 2CO.OD$

$= AO^2 + BO^2 - 2BO.OD$

$(\because BO = CO) \cdots (2)$

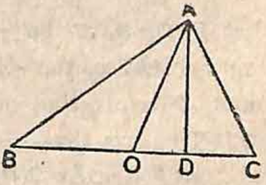


Fig. 4

Taking the difference of (1) and (2) we have,

$AB^2 \sim AC^2 = 4BO.OD = 2.(2BO).OD = 2BC.OD$ $(\because 2BO = BC)$.

Ex. 5. ABC is an isosceles triangle and AY is drawn to cut the base internally at Y. Show that $AY^2 = AB^2 - BY.YC$.

[C. U. 1919, '47]

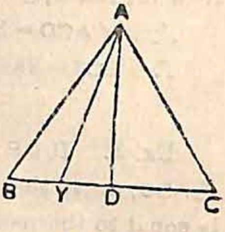


Fig. 5

In the isosceles $\triangle ABC$, $AB = AC$ and AY cuts BC at Y. Draw $AD \perp BC$.

$\because$ $\triangle ABC$ is isosceles, $\therefore$ AD bisects BC.

$\because$ $\angle D$ is a right angle, $\therefore$ $\angle B$ is acute and BD is the projection of AB on BC.

$\therefore$ $AY^2 = AB^2 + BY^2 - 2BY.BD$

$= AB^2 - BY(2BD - BY)$

$= AB^2 - BY(BC - BY) = AB^2 - BY.YC$.

Ex. 6. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

[C. U. 1931]

Let the diagonals AC, BD of the parallelogram ABCD intersect at O.

To prove that, $AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$.

Proof : $\because$ The diagonals of a parallelogram bisect each other, $\therefore$ $AO = CO$, $BO = DO$.

$\therefore$ In $\triangle ABC$, $AB^2 + BC^2 = 2AO^2 + 2BO^2$,

and in $\triangle ADC$, $AD^2 + CD^2 = 2AO^2 + 2DO^2$,

$$\begin{aligned}\therefore\ AB^2 + BC^2 + CD^2 + DA^2 &= 4AO^2 + 2BO^2 + 2DO^2\\ &= 4AO^2 + 4DO^2 \quad (\because BO = DO)\\ &= (2AO)^2 + (2DO)^2 = AC^2 + BD^2.\end{aligned}$$

Ex. 7. In any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the joins of the middle points of the opposite sides.

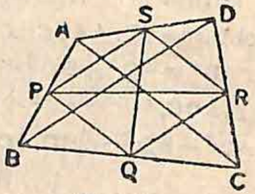


Fig. 6

Let P, Q, R, S be respectively the mid points of the sides AB, BC, CD and DA of the quadrilateral ABCD. To prove that

$$AC^2 + BD^2 = 2(PR^2 + QS^2).$$

Proof : Join AC, PR, QS, PQ, QR, RS, SP. In $\triangle ABC$, P and Q are the mid points of AB. and BC.

$\therefore$ $AC = 2PQ$. $\therefore$ $AC^2 = 4PQ^2$. Similarly $BD^2 = 4PS^2$.

Again, $\because$ PQRS is a parallelogram,

$\therefore$ $PR^2+QS^2=PQ^2+QR^2+RS^2+SP^2=2PQ^2+2PS^2$.

$\therefore$ $AC^2+BD^2=4PQ^2+4PS^2=2(2PQ^2+2PS^2)=2(PR^2+QS^2)$.

Ex. 8. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the line joining the middle points of the diagonals. [C. U. 1924]

Let E and F be the mid points of the diagonals AC and BD respectively of the quadrilateral ABCD.

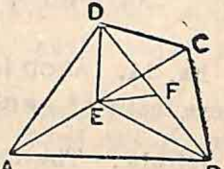


Fig. 7

To prove that $AB^2+BC^2+CD^2+DA^2$
$=AC^2+BD^2+4EF^2$.

Join EF, EB, ED.

Proof : $\because$ BE is a median of $\triangle ABC$,

$\therefore$ $AB^2+BC^2=2AE^2+2BE^2$.

Similarly in $\triangle ADC$, DE is a median

$\therefore$ $AD^2+DC^2=2AE^2+2DE^2$.

Again, $\because$ EF is a median of $\triangle EBD$,

$\therefore$ $EB^2+ED^2=2BF^2+2EF^2$.

$\therefore$ $AB^2+BC^2+CD^2+AD^2=2AE^2+2BE^2+2AE^2+2DE^2$ $=4AE^2+2(BE^2+DE^2)=4AE^2+2(2BF^2+2EF^2)=4AE^2$ $+4BF^2+4EF^2=(2AE)^2+(2BF)^2+4EF^2=AC^2+BD^2+4EF^2$.

Ex. 9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians. [C. U, '33, ,37 ; H. S. '60]

AD, BE and CF are the medians of $\triangle ABC$.

To prove that $3(AB^2+BC^2+AC^2)=4(AD^2+BE^2+CF^2)$.

Proof : $\because$ AD is a median, $\therefore$ $AB^2+AC^2=2BD^2+2AD^2$

Similarly, $AB^2+BC^2=2CE^2+2BE^2$

and $BC^2+AC^2=2AF^2+2CF^2$

(Adding), $2(AB^2+BC^2+AC^2)=2BD^2+2CE^2$ $+2AF^2+2(AD^2+BE^2+CF^2)$, $\therefore$ $4(AB^2+BC^2+AC^2)=4BD^2+$ $4CE^2+4AF^2+4(AD^2+BE^2+CF^2)=(2BD)^2+(2CE)^2+(2AF)^2+$ $4(AD^2+BE^2+CF^2)=BC^2+AC^2+AB^2+4(AD^2+BE^2+CF^2)$.

$\therefore$ $3(AB^2+BC^2+AC^2)=4(AD^2+BE^2+CF^2)$.

Ex. 10. If G be the centroid of the $\triangle ABC$, show that $AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2)$.

[As in Ex. 9 above] $3(AB^2+BC^2+AC^2)=4(AD^2+BE^2+CF^2)$ $=4\{(\frac{3}{2}AG)^2+(\frac{3}{2}BG)^2+(\frac{3}{2}CG)^2\}$ [$\because$ the medians intersect at a point of trisection] $=4(\frac{9}{4}AG^2+\frac{9}{4}BG^2+\frac{9}{4}CG^2)$

$$=9(AG^2+BG^2+CG^2).$$

$\therefore$ dividing both sides by 3, we have

$$AB^2+BC^2+AC^2=3(GA^2+GB^2+GC^2).$$

Ex. 11. $ABCD$ is a rectangle and P any point within it. Prove that $PA^2+PC^2=PB^2+PD^2$. [C. U. '27]

[**Hints** :—The diagonals intersect at O. The diagonals of a rectangle are equal and bisect each other, $\therefore$ $AO=DO$. Join PO, AP, PC, BP, DP. In $\triangle APC$, $AP^2+PC^2=2AO^2+2PO^2$. In $\triangle PDB$, $PB^2+PD^2=2DO^2+2PO^2$

$$=2AO^2+2PO^2 \ (\because AO=DO).$$

$\therefore$ $PA^2+PC^2=PB^2+PD^2$].

Ex. 12. P and Q are two points on the diameter AB of a circle, equidistant from the centre ; if C is any point on the circumference show that $PC^2+QC^2=AP^2+AQ^2$.

[**Hints** :— $\because$ $PO=QO$, $\therefore$ CO is a median of $\triangle PCQ$.

$\therefore$ $PC^2+QC^2=2CO^2+2PO^2$
$=2AO^2+2PO^2$ ($\because$ $AO=CO$).

Again, AP^2+AQ^2
$=(AO-PO)^2+(AO+OQ)^2$
$=(AO-PO)^2+(AO+PO)^2$
$=AO^2+PO^2-2AO.PO+AO^2+PO^2$
$+2AO.PO=2AO^2+2PO^2$.

$\therefore$ $PC^2+QC^2=AP^2+AQ^2$.]

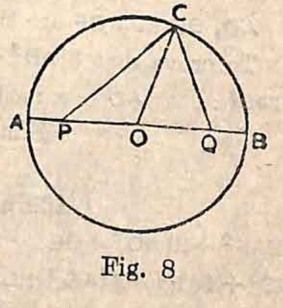


Fig. 8

Exercise 1

1. If a straight line is bisected, prove that its projection is also bisected.

2. The projections of two equal and parallel st. lines on the same st. line are equal.

3. In a $\triangle ABC$, AD is drawn perpendicular to BC; if $AD^2 = BD.CD$, prove that $\angle BAC$ is a right angle. [B. U. '14]

[**Hints** : $AB^2 + AC^2 = BD^2 + AD^2 + CD^2 + AD^2 (\because \angle D$ is a right $\angle)$

$$= BD^2 + CD^2 + 2AD^2 = BD^2 + CD^2 + 2BD.CD$$

$$= (BD + CD)^2 = BC^2, \quad \therefore \quad \angle BAC \text{ is a right angle.}]$$

4. The $\angle ACB$ of the $\triangle ABC$ is 120°, show that $AB^2 = AC^2 + BC^2 + AC.BC$.

5. A point moves so that the sum of the squares on its distances from two fixed points is constant. Find the locus of the point.

6. A straight line AB is bisected at C and produced to D. Prove that $AD^2 - BD^2 = 4AC.CD$.

7. In $\triangle ABC$, if $AB = AC$ and BD is perpendicular to AC, show that $BC^2 = 2AC.CD$.

8. Calculate the base of a triangle whose sides are 4 and 5 inches and whose median is 3·5 inches. [Ans. 5·7"].

9. Prove that the area of the triangle ABC is

$$\sqrt{s(s-a)(s-b)(s-c)} \text{ where } 2s = a + b + c.$$

10. The sides of a triangle are 6, 7 and 8 cms., show that it is an acute-angled triangle.

11. ABC is an isosceles triangle and P is any point on the base BC. Prove that $AB^2 - AP^2 = BP.PC$.

12. In $\triangle ABC$ the angles B and C are acute. If BE and CF are drawn perpendiculars to AC and AB respectively, prove that $BC^2 = AB.BF + AC.CE$. [C. U. '45]

Ratio and Proportion

58. Ratio : The ratio of one magnitude to another of the same kind is the quotient when the numerical measure of the first is divided by that of the second, the same unit measuring both of them.

Suppose there are two straight lines 6 cms. and 4 cms. in length. The ratio of their lengths is expressed by the fraction $\frac{6}{4}$ or $\frac{3}{2}$. Similary if of two quantities of the same kind one contains a units and the other b units, their ratio is $\frac{a}{b}$. This ratio may be written as $\frac{a}{b}$, $a \div b$, a/b, and generally as $a : b$.

Here a and b are the two terms of the ratio. In the ratio $a : b$, the first term a is called the *antecedent* and the second term b is called the *consequent.*

59. Commensurable *and* **incommensurable** : Magnitudes are said to be commensurable when each of them can be expressed as a multiple of a *common measure* called the *unit.* Thus 3 cms. and 5 cms. are commensurable magnitudes, because they are 3 times and 5 times the unit *one centimetre.*

Magnitudes which have no common measure are said to be incommensurable. Thus **3** and $\sqrt{2}$ are incommensurable as there is no number of which both 3 and $\sqrt{2}$ are multiples.

60. Proportion : If four magnitudes be such that the ratio of the first to the second is equal to the ratio of the third to the fourth, then they are said to be *in proportion* or *proportionals.*

Thus, if $a : b = c : d$, then a, b, c and d are proportionals. $a : b = c : d$ (or, $a : b :: c : d$) is read "a is to b as c is to d".

If a, b, c, d be in proportion, a and d are called the *extremes* and b and c are called the *means.* Here the fourth term d is called a *fourth proportional* to a, b and c.

If three magnitudes be such that the ratio of the first to the second is equal to the ratio of the second to the third, then the three magnitudes are said to be in proportion or proportionals.

Thus if $a : b = b : c$, then a, b and c are in proportion.

If a, b, c be in proportion, the second term b is called the *mean proportional* between a and c, and c is called the third proportional to a and b. Here $b^2 = ac$.

N. B. All the four magnitudes in proportion may be of the same kind or the first two may be of one same kind and the other two may be of another same kind.

Thus, 12 cm : 6 cm. = 2 : 1, and 4 seers : 2 seers = 2 : 1, and therefore 12 cm., 6 cm., 4 seers and 2 seers are proportional.

61. Some propositions regarding proportionals

(1) If $a : b = c : d$ then $\frac{a}{c} = \frac{b}{d}$ [*Alternendo*]

(2) If $a : b = c : d$, then $b : a = d : c$ [*Invertendo*]

(3) If $a : b = c : d$, then $\frac{a+b}{b} = \frac{c+d}{d}$ [*Componendo*]

(4) If $a : b = c : d$, then $\frac{a-b}{b} = \frac{c-d}{d}$ [*Dividendo*]

(5) If $a : b = c : d$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ [*Componendo & Dividendo*]

(6) If $a : b = c : d = e : f = \cdots\cdots$,

then each $= \frac{a+c+e+\cdots}{b+d+f+\cdots}$ [*Addendo*]

(7) If $a : b = c : d$, then $ad = bc$ [*cross multiplication*]

(8) If $\frac{a}{m} = \frac{b}{m}$, then $a = b$.

Theorem 3

A straight line drawn parallel to one side of a triangle cuts the other two sides, or those sides produced, proportionally.
[C. U., D. B., G. U.]

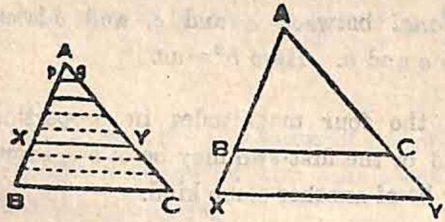


Fig. 9

Let ABC be a triangle and let the st. line XY be drawn parallel to BC to cut AB and AC or these sides produced at X and Y respectively.

To prove that AX : BX = AY : CY.

Proof : Let AX and BX be commensurable and let them have the common measure p.

Suppose AX $= m$ times $p = mp$, and BX $= n$ times $p = np$.

$$\therefore \quad \frac{AX}{BX} = \frac{mp}{np} = \frac{m}{n}.$$

Now, divide AX and BX into m and n number of equal parts, each equal to p.

Through each point of division draw straight lines parallel to BC.

Then these parallel lines will divide AY and CY into m and n number of equal parts respectively. Let each of these parts be equal to q units.

$$\text{Then } \frac{AY}{CY} = \frac{m.q}{n.q} = \frac{m}{n}.$$

$$\text{Hence } \frac{AX}{BX} = \frac{AY}{CY}, \text{ that is, } AX : BX = AY : CY.$$

[Cor. $\because \dfrac{AX}{BX}=\dfrac{AY}{CY}$, $\therefore \dfrac{AX+BX}{BX}=\dfrac{AY+CY}{CY}$, or, $\dfrac{AB}{BX}=\dfrac{AC}{CY}$.

Similarly, $\dfrac{AB}{AX}=\dfrac{AC}{AY}$ and $\dfrac{BX}{AX}=\dfrac{CY}{AY}$.]

Theorem 4

If a straight line cuts two sides of a triangle proportionally, it is parallel to the third side. [C. U. '34]

Let ABC be a triangle and let the straight line XY cut AB and AC or these sides produced at X and Y respectively so that AX : BX = AY : CY.

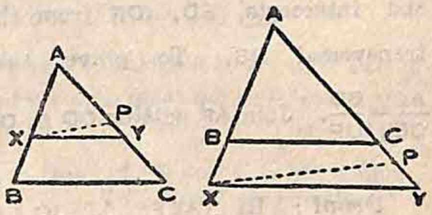


Fig. 10

To prove that XY is parallel to BC.

Proof : If XY is not parallel to BC, let XP be drawn parallel to BC cutting AC at P.

$\because$ XP ∥ BC, $\therefore \dfrac{AX}{BX}=\dfrac{AP}{PC}$;

But $\dfrac{AX}{BX}=\dfrac{AY}{CY}$ [hyp.], $\therefore \dfrac{AP}{PC}=\dfrac{AY}{CY}$,

$\therefore \dfrac{AP \pm PC}{PC}=\dfrac{AY \pm CY}{CY}$ [By componendo in the first fig. and by dividendo in the second fig.]

$\therefore \dfrac{AC}{PC}=\dfrac{AC}{CY}$, $\therefore$ PC = CY, $\therefore$ P and Y must coincide,

i.e. XP coincides with XY. $\therefore$ XY is parallel to BC.

[Cor. If $\dfrac{AX}{BX}=\dfrac{AY}{CY}$, or if $\dfrac{AB}{BX}=\dfrac{AC}{CY}$, or if $\dfrac{AB}{AX}=\dfrac{AC}{AY}$, then XY ∥ BC.]

Miscellaneous Solutions (2)

Ex. 1. Prove that three parallel straight lines cut any two transversals proportionally.

[C. U. '15, '26, '39, '40 ; G. U. '49]

Let the parallel st. lines AB, CD and EF cut off the intercepts AC, CE from the transversal PQ and the intercepts BD, DF from the transversal RS. To prove that $\frac{AC}{CE}=\frac{BD}{DF}$. Join AF cutting CD at O.

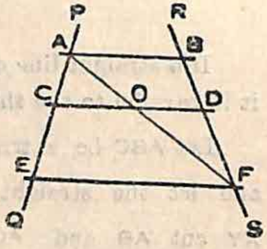


Fig. 11

Proof : In $\triangle AEF$, $\because CO \parallel EF$, $\therefore \frac{AC}{CE}=\frac{AO}{OF}$.

Again, in $\triangle ABF$, $\because OD \parallel AB$, $\therefore \frac{AO}{OF}=\frac{BD}{DF}$. $\therefore \frac{AC}{CE}=\frac{BD}{DF}$.

Ex. 2. Show that the straight line drawn through the middle point of one side of a triangle parallel to the base bisects the other side.

[C. U. '23]

In $\triangle ABC$, let X be the middle point of AB and XY be drawn parallel to BC, cutting AC at Y. To prove that AY = CY.

Proof : $\because XY \parallel BC$, $\therefore \frac{AX}{BX}=\frac{AY}{CY}$, but AX = BX (hyp.),

$$\therefore AY = CY.$$

Ex. 3. Show that the straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

[D. B. '37, '41, '44]

Let P and Q be the mid points of the oblique sides AC and BD respectively of the trapezium ABDC. Join PQ.

To prove that PQ is parallel to the parallel sides AB and CD.

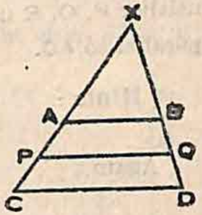


Fig. 12

Produce CA, DB to meet at X.

Proof: In $\triangle XCD$, $AB \parallel CD$,

$\therefore \frac{XA}{AC}=\frac{XB}{BD}$, or, $\frac{XA}{2AP}=\frac{XB}{2BQ}$, or, $\frac{XA}{AP}=\frac{XB}{BQ}$,

$\therefore$ $AB \parallel PQ$. Again, $\because$ $AB \parallel CD$,

$\therefore$ $PQ \parallel CD$. $\therefore$ PQ is parallel to AB and CD.

Ex. 4. In any triangle ABC, AB is bisected in D and CD is bisected in E. If AE is produced to meet BC in F, prove that $FC=\frac{1}{3}BC$. [D. B. '39]

In $\triangle ABC$, D is the middle point of AB and E is the mid pt. of CD. AE produced meets BC in F. To prove that $FC=\frac{1}{3}BC$.

Draw $DR \parallel AF$, cutting BC at R.

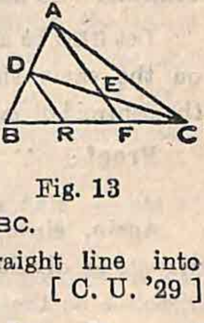


Fig. 13

Proof: In $\triangle ABF$, $\because$ $DR \parallel AF$,

$\therefore \frac{BD}{DA}=\frac{BR}{RF}$; but $BD=DA$, $\therefore$ $BR=RF$.

Again, in $\triangle CDR$, $\because$ $EF \parallel DR$,

$\therefore$ $CE : DE = CF : RF$, but $CE=DE$,

$\therefore$ $CF=RF$. $\therefore$ $CF=RF=BR$, $\therefore$ $FC=\frac{1}{3}BC$.

Ex. 5. Show how to divide a given straight line into three equal parts. [C. U. '29]

[**Hints**; Let AB be the given straight line. Draw AX making an angle with AB. From AX cut off 3 equal parts AP, PQ, QR of any length. Join RB and from P, Q draw straight lines parallel to RB cutting AB at C and D respectively. Then AB is divided into 3 equal parts at C and D. $\because$ $PC \parallel RB$, $\therefore$ $AC : AB = AP : AR = 1 : 3$. $\therefore$ $AC=\frac{1}{3}AB$. Similarly CD and DB are each equal to $\frac{1}{3}AB$.]

Ex. 6. In three straight lines OPA, OQB, ORC the points are so chosen that PQ is parallel to AB, QR is parallel to BC, neither P, Q, R nor A, B, C are collinear. Prove that PR is parallel to AC. [C. U. '47]

[**Hints** : $\because$ PQ $\parallel$ AB, $\therefore$ $\frac{OP}{AP}=\frac{OQ}{BQ}$.

Again, $\because$ QR $\parallel$ BC, $\therefore$ $\frac{OQ}{BQ}=\frac{OR}{CR}$.

$\therefore$ $\frac{OP}{AP}=\frac{OR}{CR}$. Now, in $\triangle$OAC,

$\because$ $\frac{OP}{AP}=\frac{OR}{CR}$, $\therefore$ PR $\parallel$ AC.]

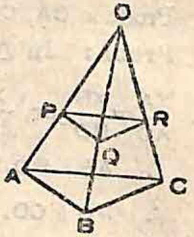


Fig. 14

Ex. 7. Find the locus of the centroid of a triangle standing on a given base and having a given area. [C.U.'34,'43,'42,'45]

Let the base BC and the area of the $\triangle$ABC be equal to the given base and the given area. Let AD be its median and G the centroid. To find the locus of G.

Let EBC be another triangle standing on the same base BC on the same side of it, and equal in area to $\triangle$ABC. Let H be the centroid of $\triangle$EBC. Join GH and AE.

Proof : $\because$ $\triangle$ABC$=\triangle$EBC,

$\therefore$ AE $\parallel$ BC.

Again, since each median of a triangle is divided at a point of trisection at the centroid,

$\therefore$ $\frac{DG}{DA}=\frac{1}{3}$ and $\frac{DH}{DE}=\frac{1}{3}$,

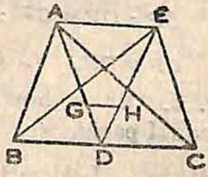


Fig. 15

$\therefore$ $\frac{DG}{DA}=\frac{DH}{DE}$, $\therefore$ GH $\parallel$ AE. $\therefore$ GH $\parallel$ BC.

Hence at any position of the vertex A of the $\triangle$ABC, its centroid must lie on GH. $\therefore$ the straight line drawn through G parallel to BC is the locus of the centroid G.

Ex. 8. OPQ is a straight line drawn to a fixed point O and is such that $OP : OQ$ is constant. If P moves along a fixed straight line, find the locus of Q. [C. U. '27]

Let O be the given fixed point and the straight line OPQ be drawn so that $OP : OQ$ is constant (*i.e.*, always the same). P always moves along the fixed straight line PX.

To find the locus of Q.

Through Q draw $QY \parallel PX$. Let the point A on the straight line PX be one of the positions of P. Join OA and produce it to cut QY at B. Now, $\because PA \parallel QB$,

$\therefore OA : OB = OP : OQ =$ constant.

So, if P be at A, Q must be at B on QY.

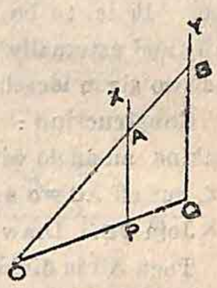


Fig. 16

Hence the locus of Q is the straight line drawn through Q parallel to PX.

Exercise 2

1. In $\triangle ABC$, PQ is drawn parallel to BC cutting the other sides produced at P and Q. If $AB = 4{\cdot}2$ cm., $AC = 3{\cdot}6$ cm., and $AP = 6{\cdot}3$ cm., find the length of AQ. [Ans. 5·4 cm.]

2. A series of parallel st. lines cuts any two transversals proportionally.

3. The straight line which joins the middle points of two sides of a triangle is parallel to the third side.

4. In $\triangle ABC$, E is the mid point of the median AD and BE produced cuts AC in F ; prove that $AF = \frac{1}{3}AC$.

5. P and Q are any two points on two parallel st. lines. PQ is divided at X in a given ratio. Find the locus of the point X.

6. Two triangles ABC, DBC stand on the same side of the common base BC, and from any point E in BC lines are drawn parallel to BA, BD meeting AC, DC in F and G. Show that FG is parallel to AD. [G. U. '51]

Problem 1

To divide a given straight line internally and externally in the ratio of two given straight lines.

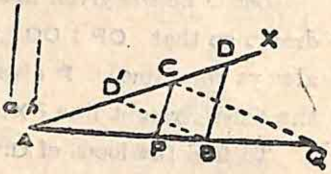


Fig. 17

Let AB be the given straight line. It is to be divided internally and externally in the ratio of the two given lengths a and b.

Construction : Draw AX making an angle with AB. From AX, cut off $AC=a$ and $CD=b$. From CA cut off $CD'=b$.

Join DB. Draw $CP \parallel DB$ cutting AB at P.

Then AB is divided internally at P in the ratio $a : b$.

Again, join D′B and drow $CQ \parallel D'B$ cutting AB produced at Q. Then AB is divided externally at Q in the ratio $a : b$.

Proof : In $\triangle ABD$, $\because CP \parallel DB$, $\therefore AP : PB = AC : CD = a : b$.

Again, in $\triangle ABD'$, $\because CQ \parallel D'B$, $\therefore AQ : QB = AC : CD' = a : b$

[**N. B.** Thus a straight line can be divided at any ratio. If the given ratio be $3 : 2$, take $AC = 3$ units of length and $CD = 2$ units of length.]

Problem 2

To construct the fourth proportional to three given straight lines.

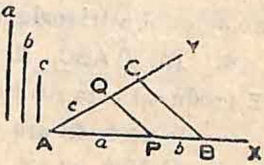


Fig. 18

Let a, b, c be the three given straight lines.

To construct the fourth proportional to a, b, c.

Construction : Take any straight line AX. From AX, cut off $AP=a$ and $PB=b$. From A draw AY making an angle with AX. From AY, cut off $AQ=c$. Join PQ and draw $BC \parallel PQ$, cutting AY at C.

Then QC is the required fourth proportional.

Proof : $\because$ PQ $\parallel$ BC, $\therefore$ AP : PB = AQ : QC,
that is, $a : b = c$: QC.

$\therefore$ QC is the fourth proportional to a, b, c.

Problem 3

To find a third proportional to two given straight lines.

Let a and b be the two given straight lines. It is required to find the third proportional to a and b.

Construction : Draw any two straight lines AX, AY making an angle XAY. From AX, cut off AP $= a$ and PB $= b$, and from AY, cut off AQ $= b$. Join PQ. Through B, draw BC parallel to PQ, cutting AY at C. Then CQ is the required third proportional.

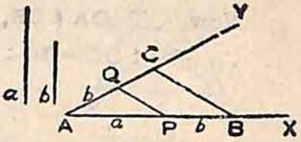


Fig. 19

Proof : $\because$ PQ $\parallel$ BC,

$\therefore$ AP : BP = AQ : QC, that is, $a : b = b$: CQ

$\therefore$ CQ is the third proportional to a and b.

Theorem 5

The internal or external bisector of an angle of a triangle divides the opposite side internally or externally in the ratio of the other two sides. [C. U. '39, '41, '44, '46, '49 ; D. B. '48, '51 ; G. U. '48, '50, '52 ; W. B. S. B. '52]

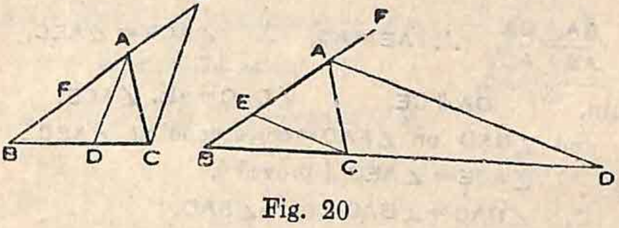


Fig. 20

Let ABC be a triangle and AD, the internal bisector of $\angle$BAC, cut BC internally at D and let its external bisector AD cut BC produced at D.

To prove that BD : DC = AB : AC.

Construction : From C, draw CE parallel to DA, cutting BA or BA produced at E.

Proof : $\because$ DA $\parallel$ CE, $\therefore$ $\angle$DAC = alt. $\angle$ACE,
and $\angle$BAD or $\angle$FAD = corresponding $\angle$AEC,
but $\angle$FAD = $\angle$CAD, $\therefore$ $\angle$ACE = $\angle$AEC, $\therefore$ AC = AE.
Now, $\because$ DA $\parallel$ CE,
$\therefore$ BD : DC = BA : AE = BA : AC [$\because$ AE = AC].

Theorem 6

If a straight line drawn from an angle of a triangle divides the opposite side internally or externally in the ratio of the other two sides, then the straight line is the internal or external bisector of the angle. [C. U. '42 ; D. B. '33]

[Draw the figure of Theorem 5] Let ABC be a triangle and let AD, drawn from the $\angle$A, divide BC or BC produced at D so that BD : CD = AB : AC.

To prove that AD is the internal or external bisector of the $\angle$BAC.

Construction : Through C, draw CE $\parallel$ DA, meeting BA or BA produced at E.

Proof : $\because$ DA $\parallel$ CE, $\therefore$ $\frac{BD}{DC} = \frac{BA}{AE}$; but $\frac{BD}{DC} = \frac{BA}{AC}$ (hyp.)

$\therefore$ $\frac{BA}{AE} = \frac{BA}{AC}$, $\therefore$ AE = AC, $\therefore$ $\angle$ACE = $\angle$AEC.

Again, $\because$ DA $\parallel$ CE, $\therefore$ $\angle$DAC = alt. $\angle$ACE,
and $\angle$BAD or $\angle$FAD = corresponding $\angle$AEC ;
but $\angle$ACE = $\angle$AEC (proved),
$\therefore$ $\angle$DAC = $\angle$BAD or $\angle$FAD.

$\therefore$ AD is the internal or external bisector of $\angle$BAC.

[**N. B.** If a st. line is divided internally and externally in the same ratio, it is said to be divided **harmonically.**]

Miscellaneous Solutions (3)

Ex. 1. The bisectors of the base angles of a triangle meet the opposite sides at X and Y respectively. If XY is parallel to the base, prove that the triangle is isosceles.

In $\triangle ABC$, let BY and CX bisect the $\angle B$ and $\angle C$ and meet the opposite sides AC and AB at Y and X respectively.

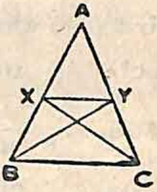


Fig. 21

Let XY be parallel to BC.

To prove that $\triangle ABC$ is isosceles.

Proof : $\because$ BY is the bisector of $\angle ABC$, $\therefore \frac{AB}{BC}=\frac{AY}{CY}$.

$\because$ CX bisects $\angle ACB$, $\therefore \frac{AC}{BC}=\frac{AX}{BX}$.

Again, $\because XY \parallel BC$ (hyp.), $\therefore \frac{AX}{BX}=\frac{AY}{CY}$. $\therefore \frac{AB}{BC}=\frac{AC}{BC}$,

$\therefore AB=AC$. $\therefore$ the $\triangle ABC$ is isosceles.

Ex. 2. Prove that the internal bisectors of the angles of a triangle meet at a point. [C.U. '14 ; W.B.S.F. '52 ; D.B. '44]

Let BI and CI, the internal bisectors of $\angle B$ and $\angle C$ respectively of $\triangle ABC$, meet at I. Join AI.

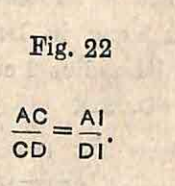


Fig. 22

To prove that AI is the bisector of $\angle A$.

Produce AI to meet BC at D.

Proof : In $\triangle ABD$, $\because$ BI bisects $\angle B$,

$\therefore \frac{AB}{BD}=\frac{AI}{DI}$.

In $\triangle ADC$, $\because$ CI is the bisector of $\angle C$, $\therefore \frac{AC}{CD}=\frac{AI}{DI}$.

$\therefore \frac{AB}{BD}=\frac{AC}{CD}$, $\therefore \frac{AB}{AC}=\frac{BD}{CD}$. $\therefore$ AI is the bisector of $\angle A$.

Ex. 3. AD is a median of the $\triangle ABC$; and the angles ABD and ADC are bisected by lines which meet AB, AC at E and F respectively. Show that EF is parallel to BC.

[C.U. '37, '39, '42 ; D.B. '42, '51 ; G. U. '48]

[First state what is given] Join EF.

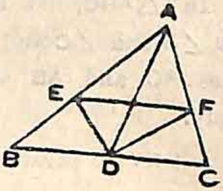


Fig. 23

Proof : $\because$ DE is the bisector of

$\angle ADB$, $\therefore$ $\dfrac{AE}{BE}=\dfrac{AD}{BD}$.

Again, $\because$ DF bisects $\angle ADC$,

$\therefore$ $\dfrac{AF}{CF}=\dfrac{AD}{DC}=\dfrac{AD}{BD}$ ($\because$ BD = DC),

$\therefore$ $\dfrac{AE}{BE}=\dfrac{AF}{FC}$, $\therefore$ EF $\parallel$ BC.

Ex. 4. Draw a straight line AB of length 4 cm. and divide it externally at C in the ratio 5 : 2. [E. B. S. B. '49]

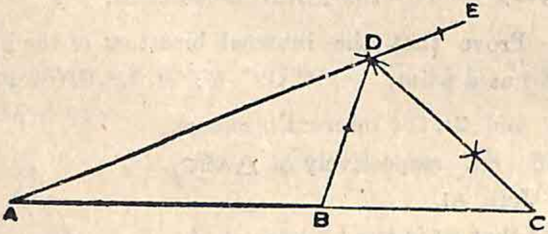


Fig. 24

Take a straight line AB = 4 cm. With centre A and with radius 5 cm. in length draw an arc of a circle. With centre B and with radius 2 cm. draw an arc, cutting the former arc at D. Join AD, BD.

Draw DC the external bisector of $\angle ADB$ (*i.e.*, the internal bisector of $\angle BDE$) and let it cut AB produced at C. Then AB is divided externally at C in the ratio 5 : 2.

Proof : ∵ DC is the external bisector of $\angle ADB$;

$\therefore$ $AC : BC = AD : DB = 5 : 2.$

[**N. B.** It can be easily done as in problem 1.]

Ex. 5. ABCD is a quadrilateral. Show that if the bisectors of the angles A and C meet on the diagonal BD, the bisectors of the angles B and D will meet on AC.

[C. U. '12, '38 ; G. U. '50]

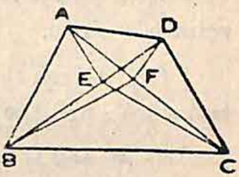


Fig. 25

In the quadrilateral ABCD, let the bisectors of $\angle A$ and $\angle C$ meet at E on the diagonal BD. Let BF, the bisector of $\angle B$, meet the diagonal AC at F. Join DF. To prove that DF is the bisector of $\angle ADC$.

Proof : In $\triangle ABD$, AE bisects $\angle BAD$, $\therefore$ $AB : AD = BE : DE.$ In $\triangle BCD$, CE is the bisector of $\angle BCD$,

$\therefore$ $BC : DC = BE : DE.$ $\therefore$ $\dfrac{AB}{AD} = \dfrac{BC}{DC}$, $\therefore$ $\dfrac{AB}{BC} = \dfrac{AD}{DC}$,

Again, ∵ BF is the bisector of $\angle ABC$, $\therefore$ $\dfrac{AB}{BC} = \dfrac{AF}{CF}$.

$\therefore$ $\dfrac{AD}{DC} = \dfrac{AF}{CF}$, $\therefore$ DF is the bisector of $\angle ADC$.

Ex. 6. The bisectors of the angles of the $\triangle ABC$ intersect the sides BC, CA, AB at L, M, N respectively. Prove that $BL.CM.AN = LC.MA.NB$.

[First state what is given]

Proof : ∵ AL is the bisector of $\angle A$,

$\therefore$ $\dfrac{BL}{LC} = \dfrac{AB}{AC} \cdots\cdots(1)$

∵ BM is the bisector of $\angle B$,

$\therefore$ $\dfrac{CM}{AM} = \dfrac{BC}{AB} \cdots\cdots(2)$

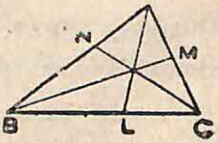


Fig. 26

Similarly, $\dfrac{AN}{BN} = \dfrac{AC}{BC} \cdots\cdots(3)$

Now, taking the product of (1), (2) and (3) we have

$$\frac{BL}{CL}\cdot\frac{CM}{AM}\cdot\frac{AN}{BN}=\frac{AB}{AC}\cdot\frac{BC}{AB}\cdot\frac{AC}{BC}=1.$$

$\therefore$ BL.CM.AN=LC.MA.NB.

Ex. 7. A point moves so that the ratio of its distances from two fixed points is constant, prove that the locus of the moving point is a circle. [C.U. '44, '49 ; D.B. '33, '46]

[Or, Given the base of a triangle and the ratio of the other two sides ; find the locus of the vertex.]

Let B and C be two fixed points and let the point A move so that AB : AC is always constant. To prove that the locus of A is a circle.

Let A be any position of the moving point. Draw AE and AF, the internal and external bisectors of ∠BAC, meeting BC in E and F respectively.

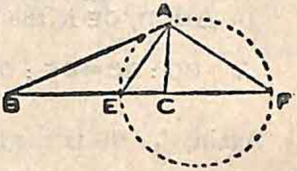


Fig. 27

Proof : $\because$ AE and AF are bisectors of ∠BAC, $\therefore$ $\frac{BE}{CE}=\frac{BF}{CF}=\frac{AB}{AC}=$ constant.

$\therefore$ BC is divided at E and F in a constant ratio.

Since B, C are fixed points, $\therefore$ E and F are fixed points.

$\therefore$ EF is a straight line of fixed length.

Again, ∠EAF, being the angle between the internal and external bisectors of ∠BAC, is a right angle.

$\therefore$ the fixed straight line EF always subtends a right angle at A. Hence, the locus of A is a circle described on EF as diameter.

[N. B. This circle is known as the *circle of Apollonius.*]

Ex. 8. Construct a triangle having given the base, the vertical angle and the ratio of the other two sides.

Let BC be the given base, ∠A the given vertical angle and let $m : n$ be the given ratio so that AB : AC $= m : n$.

To construct the triangle.

Construction : Divide BC internally at E in the ratio $m : n$.

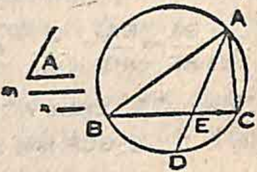


Fig. 28

On BC draw a segment of a circle containing ∠A and complete the circle. Bisect the conjugate arc of arc BAC at D.

Join DE and produce it to meet the circumference at A.

Join AB and AC. Then ABC is the triangle required.

Proof : ∵ arc BD = arc CD, ∴ ∠BAD = ∠CAD.

∴ AE is the bisector of ∠BAC.

∴ AB : AC = BE : EC $= m : n$,

and by construction ∠BAC = the given ∠A.

Ex. 9. If a straight line BC is divided harmonically at D and E and if DE subtends a right angle at any point A outside BC, then AD and AE are the internal and external bisectors of the angle between AB and AC. [*cf.* H. S. '60]

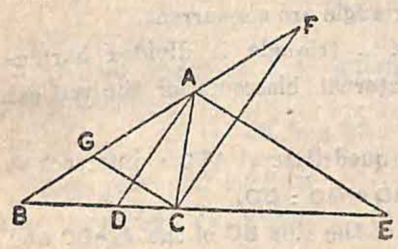


Fig. 29

Let the straight line BC be divided internally at D and externally at E in the same ratio $m : n$ and let DE subtend a right angle DAE at a point A.

To prove that AD and AE are respectively the internal and external bisectors of ∠BAC.

Construction : From C, draw CG ∥ EA and CF ∥ DA, and let CG and CF cut BA and BA produced at G and F respectively.

Proof : $\because$ in $\triangle BCF$, $DA \parallel CF$, $\therefore$ $BA : AF = BD : CD$.

Again, in $\triangle BGC$, $\because$ $AE \parallel GC$, $\therefore$ $BE : CE = BA : GA$;

but $BD : DC = BE : CE$ (hyp.),

$$\therefore \frac{BA}{AF} = \frac{BA}{AG} \quad \therefore \quad AF = AG.$$

Now, $\because$ figure AC is a parallelogram and $\angle DAE$ is a right angle, $\therefore$ $\angle GCF$ is a right angle.

In the right-angled $\triangle GCF$, A is the mid point of the hypotenuse GF, $\therefore$ $GA = AF = AC$.

$\therefore$ $\angle AGC = \angle ACG =$ alternate $\angle CAE$.

Again, $\because$ $CG \parallel EA$, $\therefore$ $\angle FAE =$ corresponding $\angle AGC$.

$\therefore$ $\angle FAE = \angle CAE$. $\therefore$ AE is the bisector of $\angle CAF$,

i.e., AE is the external bisector of $\angle BAC$.

Again, $\angle GAD = \angle AFC = \angle ACF$ ($\because$ $AC = AF$) $=$ alt. $\angle CAD$;

$\therefore$ AD is the internal bisector of $\angle BAC$.

Hence AD and AE are respectively the internal and the external bisectors of $\angle BAC$.

Exercise 3

1. Apply theorem 5 to trisect a given straight line.

2. The external bisectors of two angles and the internal bisector of the third angle of a triangle are concurrent.

3. Prove that the base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle.

4. The diagonals of a cyclic quadrilateral ABCD intersect at O ; if $BC = CD$, show that $BA : AD = BO : DO$.

5. X is the middle point of the side BC of the $\triangle ABC$ and the angles AXB, AXC are bisected externally by the lines XE, XF which meet AB and AC produced at E and F respectively. Prove that $EF \parallel BC$. [C.U. '39]

6. If I is the in-centre of the $\triangle ABC$ and AI produced meets BC in D, then $AI : ID = AB + AC : BC$.

7. The two bisectors of the $\angle A$ of the $\triangle ABC$ cut BC in P and Q. If X is the mid pt. of BC, then $XC^2 = PX.QX$.

8. In an isosceles $\triangle ABC$ of which each of the base angles B and C is double of the $\angle A$, the bisector of the angle C meets AB at D. Prove that AB, BC and BD are three proportionals.

9. If a straight line AB is divided internally at C and externally at D in the same ratio $m : n$, and if Q be any point on the circle on CD as diameter, then $QA : QB = m : n$.

Similar triangles and polygons

62. Two triangles (or two polygons) are said to be **equiangular** to one another, if the angles of the one are equal to the angles of the other, when taken in the same order.

It is evident that two triangles will be equiangular, when two angles of the one are known to be equal to two angles of the other, for their third angles must then be equal.

63. Two triangles (or two polygons) are said to be **similar**, when they are equiangular and their corresponding sides are proportional.

Now, two sides of two similar triangles (or polygons) are taken to be corresponding sides, if they are opposite to two of their equal angles.

Theorem 7

If two triangles are equiangular, their corresponding sides are proportional. [C. U. '38, '41, 43, 47, '48, '51 ; G. U. '48, '50, 52 ; D. B. '40, '41, 43, 45, '50 '51]

Let ABC and DEF be two triangles, in which $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

To prove that

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

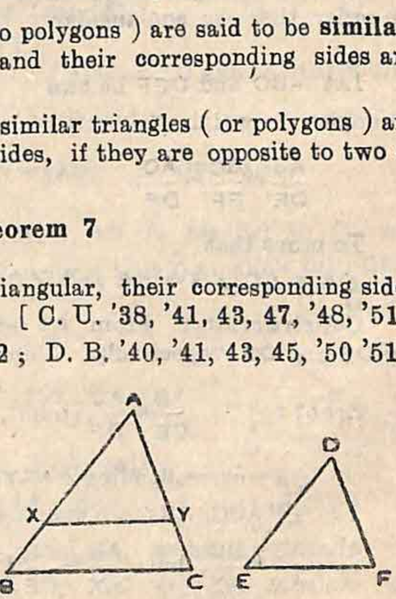


Fig. 30

Construction : From AB and AC cut off AX and AY equal to DE and DF respectively. Join XY.

Proof : In $\triangle^s$AXY and DEF, AX = DE, AY = DF, and the included $\angle$A = the included $\angle$D, $\therefore$ the triangles are congruent,

$\therefore$ $\angle AXY = \angle DEF = \angle ABC$ [hyp.], $\therefore$ XY $\parallel$ BC.

$\therefore$ $\frac{AB}{AX} = \frac{AC}{AY}$, *i. e.*, $\frac{AB}{DE} = \frac{AC}{DF}$.

Similarly by cutting off lengths from BA and BC equal to ED and EF respectively, it can be proved that $\frac{AB}{DE} = \frac{BC}{EF}$.

Hence $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

Theorem 8

(Converse of Theorem 7)

If two triangles have their sides proportional, when taken in order, they are equiangular.

[C. U. '42, 45 ; D. B. '32, '45]

Let ABC and DEF be two triangles, in which

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

To prove that

$\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$.

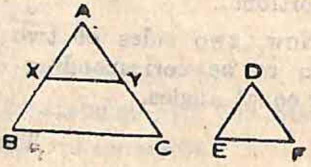


Fig. 01

Construction : From AB and AC cut off AX and AY equal to DE and DF respectively. Join XY.

Proof : $\because$ $\frac{AB}{DE} = \frac{AC}{DF}$ (hyp.), $\therefore$ $\frac{AB}{AX} = \frac{AC}{AY}$, $\therefore$ XY $\parallel$ BC.

$\therefore$ $\angle B$ = corresponding $\angle AXY$ and $\angle C$ = corresponding $\angle AYX$,

$\therefore$ $\triangle^s$ABC, AXY are equiangular,

$\therefore$ $\frac{AB}{AX} = \frac{BC}{XY}$, but $\frac{AB}{AX} = \frac{AB}{DE}$ ($\because$ AX = DE) = $\frac{BC}{EF}$ (hyp.),

$\therefore$ $\frac{BC}{XY} = \frac{BC}{EF}$, $\therefore$ XY = EF.

Now, in the $\triangle^s AXY$ and DEF, $\because$ $AX=DE$, $AY=DF$ and $XY=EF$, $\therefore$ the triangles are congruent, so that

$\angle AXY=\angle E$ and $\angle AYX=\angle F$.

$\therefore$ $\angle B=\angle AXY=\angle E$ and $\angle C=\angle AYX=\angle F$.

$\therefore$ the $\triangle^s ABC$ and DEF are equiangular.

Miscellaneous Solutions (4)

Ex. 1. The line joining the middle points of any two sides of a triangle is parallel to the third side and is half of it.

[Draw a figure] Let ABC be a triangle in which D, E are the middle points of AB, AC respectively. Join DE.

To prove that $DE \parallel BC$ and $DE=\frac{1}{2}BC$.

Proof : $\because$ D and E are the mid points of AB and AC,

$\therefore$ $\frac{AD}{DB}=\frac{AE}{EC}$, $\therefore$ $DE \parallel BC$ (proved).

Again. $\because$ $DE \parallel BC$, $\therefore$ $\angle ADE=\angle B$ and $\angle AED=\angle C$.

$\therefore$ $\triangle ADE$ and $\triangle ABC$ are equiangular.

$\therefore$ $\frac{DE}{BC}=\frac{AD}{AB}=\frac{1}{2}$ ($\because$ $AD=\frac{1}{2}AB$), $\therefore$ $DE=\frac{1}{2}BC$.

Ex. 2. In the trapezium ABCD, AB is parallel to DC and the diagonals intersect at O ; show that $OA : OC=OB : OD= AB : DC$. [C. U. '46]

Proof : $\because$ $AB \parallel DC$,

$\therefore$ $\angle BAO=$alt. $\angle OCD$ and $\angle ABO=$alt. $\angle ODC$,

$\therefore$ $\triangle AOB$ and $\triangle DOC$ are equiangular.

$\therefore$ $AO : OC=OB : OD=AB : DC$.

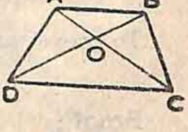


Fig. 32

Ex. 3. If through any point X within a circle two chords AB, CD are drawn and AC, BD are joined, show that $AX : DX=CX : BX$. [D. B. '40]

[Draw the figure and state what is given].

[**Hints** : $\angle A = \angle D$ being angles at the circumference on the same arc, $\angle AXC$ = vert. opp. $\angle DXB$, so $\triangle ACX$ and $\triangle DBX$ are equiangular. $\therefore AX : DX = CX : BX$.]

Ex. 4. Two circles touch one another externally at A and a straight line through A cuts them at P and Q. Prove that their diameters are as AP : AQ.

Let two circles touch externally at A and the st. line PAQ, drawn through A, cut the circles at P and Q. Let AB and AC be the diameters of the circles.

To prove that AB : AC = AP : AQ.

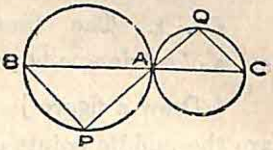


Fig. 33

Proof : Join BP and CQ. $\because$ the circles touch at A, $\therefore$ their two centres and the pt. A lie on the same straight line. $\therefore$ AB and AC are in the same st. line.

Now, the angles P and Q being in a semi-circle are each a right angle, and the $\angle BAP$ = vertically opp. $\angle QAC$.

$\therefore$ $\triangle^s APB$, AQC are equiangular. $\therefore AB : AC = AP : AQ$.

Ex. 5. Prove that the perimeters of any two similar triangles are in the ratio of any two corresponding sides.

Let ABC, DEF be two similar triangles in which AB and DE, AC and DF, BC and EF are corresponding sides.

To prove that $\dfrac{\text{the perimeter of } \triangle ABC}{\text{the perimeter of } \triangle DEF} = \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$.

Proof : $\because$ the $\triangle^s ABC$, DEF are similar,

$\therefore \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$. $\therefore$ By addendo we have

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = \frac{AB+AC+BC}{DE+DF+EF} = \frac{\text{the perimeter of } \triangle ABC}{\text{the perimeter of } \triangle DEF}.$$

Ex. 6. Two parallel straight lines meet the straight lines OP, OQ, OR at P, Q, R and P′, Q′, R′ respectively ; prove that PQ : QR = P′Q′ : Q′R′. [C. U. '41 ; E.B.S.B. '51 ; G. U. '50]

[State what is given]

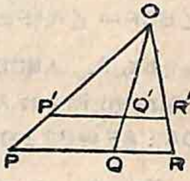


Fig. 34

Proof : $\because P'Q' \parallel PQ$,

$\therefore \angle OP'Q' = \angle P$ and $\angle OQ'P' = \angle OQP$.

$\therefore \triangle OP'Q'$ and $\triangle OPQ$ are equiangular,

$\therefore \dfrac{P'Q'}{PQ} = \dfrac{OQ'}{OQ}$.

Similarly, $\dfrac{OQ'}{OQ} = \dfrac{Q'R'}{QR}$ ($\because \triangle OQ'R'$ and $\triangle OQR$ are equiangular.)

$\therefore \dfrac{P'Q'}{PQ} = \dfrac{Q'R'}{QR}$, $\therefore$ by alternendo $\dfrac{P'Q'}{Q'R'} = \dfrac{PQ}{QR}$.

Ex. 7. Any straight line drawn parallel to the base of a triangle is bisected by the line drawn from the vertex to the middle point of the base. [C. U. '14]

[See the above fig.] Let OQ be a median of $\triangle OPR$ and let P′R′ be drawn parallel to PR, cutting OP, OQ and OR at P′, Q′, R′ respectively. To prove that $P'Q' = Q'R'$.

Proof : [First prove as in Ex. 6 above that $\dfrac{P'Q'}{Q'R'} = \dfrac{PQ}{QR}$.

Now, $\because PQ = QR$ (hyp.), $\therefore P'Q' = Q'R'$.]

Ex. 8. ABCD is a cyclic quadrilateral ; AB, DC are produced intersecting at P outside the circle. Prove that PB : PD = PC : PA. [C. U. '11, '48 ; E. B. S. B. '48]

[**Hints :** Since the side AB of the cyclic quadrilateral ABCD is produced,

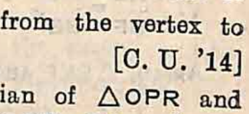


Fig. 35

$\therefore$ the ext. $\angle PBC$ = the int. opp. $\angle D$.

Now, in $\triangle^s$ BPC and APD,

$\angle PBC = \angle D$ and $\angle P$ is common to both.

So their third angles are equal.

$\therefore$ the triangles are equiangular. $\therefore \dfrac{BP}{PD} = \dfrac{PC}{PA}$.

[**N. B.** The corresponding sides of two equiangular triangles are proportional. Here $\angle PBC = \angle D$, $\therefore$ their opposite sides PC and AP are corresponding sides. As $\angle BCP = \angle A$, PB and PD are corresponding sides.]

Ex. 9. ABCD is a parallelogram. From D a straight line is drawn to cut AB at E and CB produced at F. Show that DA : AE = BF : BE = FC : CD. [C. U. '38]

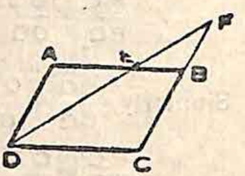


Fig. 36

Proof : $\angle AED$ = vertically opp. $\angle BEF$ and $\angle A$ = alt. $\angle EBF$; so the remaining angles are equal.

$\therefore$ $\triangle ADE$ and $\triangle BEF$ are equiangular and hence similar.

$$\therefore \frac{AD}{BF} = \frac{AE}{BE}, \quad \therefore \frac{AD}{AE} = \frac{BF}{BE} \text{ (by alternendo).}$$

Again, $\triangle BEF$ and $\triangle FDC$ are equiangular,

($\because$ $\angle FEB$ = cor. $\angle FDC$ and $\angle EBF$ = cor. $\angle C$),

$$\therefore \frac{FB}{FC} = \frac{BE}{DC}, \quad \therefore \frac{BF}{BE} = \frac{FC}{CD}. \quad \therefore \frac{AD}{AE} = \frac{BF}{BE} = \frac{FC}{CD}.$$

Ex. 10. Prove that any chord of a circle is a mean proportional between its diameter and the perpendicular from either extremity of the chord upon the tangent at the other. [C. U. '20]

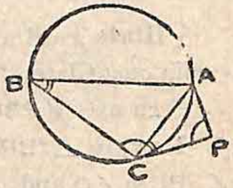


Fig. 37

Let AC be any chord of a circle and PC be a tangent to the circle at C. AP is drawn perpendicular to CP and AB is a diameter of the circle.

To prove that AB : AC = AC : AP (or, $AC^2 = AB.AP$). Join BC.

Proof : $\angle ACB$ being in a semi-circle is a right angle and $\angle P$ is also a right angle.

∵ PC is a tangent and CA is a chord through the point of contact, ∴ $\angle ACP = \angle B$ in the alternate segment.

∴ △ˢABC, APC are equiangular. ∴ $\frac{AB}{AC} = \frac{AC}{AP}$.

Ex. 11. A, B, C and D are four points lying in order on a straight line. Find a point X on it such that $XA : XB = XC : XD$.
[C. U. '42 ; E. B. S. B. '50]

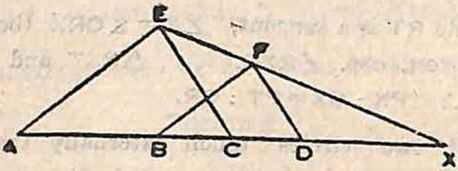


Fig. 38

Let A, B, C and D be four points on the same st. line. To find a point X on the st. line so that $XA : XB = XC : XD$.

Construction : On AC draw any △AEC. From B and D draw st. lines parallel respectively to AE and CE and let them meet at F. Join EF and produce it to meet AD produced at X. Then X is the point required.

Proof : ∵ $BF \parallel AE$, ∴ $\frac{AX}{BX} = \frac{EX}{FX}$.

Again, ∵ $DF \parallel CE$, ∴ $\frac{CX}{DX} = \frac{EX}{FX}$. ∴ $\frac{AX}{BX} = \frac{CX}{DX}$.

Ex. 12. A common tangent to two circles cuts the line joining their centres, internally or externally, in the ratio of their radii.

Let P, Q be the centres of the circles and let their common tangent TR cut PQ, the line of centres, internally (in the first fig.) or externally (in the second fig.) at X.

[**N. B.** The corresponding sides of two equiangular triangles are proportional. Here $\angle PBC = \angle D$, $\therefore$ their opposite sides PC and AP are corresponding sides. As $\angle BCP = \angle A$, PB and PD are corresponding sides.]

Ex. 9. ABCD is a parallelogram. From D a straight line is drawn to cut AB at E and CB produced at F. Show that DA : AE = BF : BE = FC : CD. [C. U. '38]

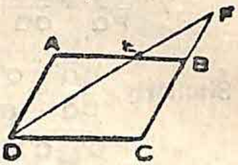


Fig. 36

Proof : $\angle AED$ = vertically opp. $\angle BEF$ and $\angle A$ = alt. $\angle EBF$; so the remaining angles are equal.

$\therefore$ $\triangle ADE$ and $\triangle BEF$ are equiangular and hence similar.

$$\therefore \frac{AD}{BF} = \frac{AE}{BE}, \quad \therefore \frac{AD}{AE} = \frac{BF}{BE} \text{ (by alternendo).}$$

Again, $\triangle BEF$ and $\triangle FDC$ are equiangular,

($\because$ $\angle FEB$ = cor. $\angle FDC$ and $\angle EBF$ = cor. $\angle C$),

$$\therefore \frac{FB}{FC} = \frac{BE}{DC}, \quad \therefore \frac{BF}{BE} = \frac{FC}{CD}. \quad \therefore \frac{AD}{AE} = \frac{BF}{BE} = \frac{FC}{CD}.$$

Ex. 10. Prove that any chord of a circle is a mean proportional between its diameter and the perpendicular from either extremity of the chord upon the tangent at the other. [C. U. '20]

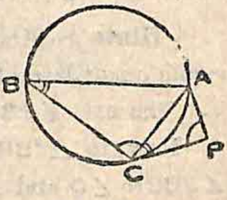


Fig. 37

Let AC be any chord of a circle and PC be a tangent to the circle at C. AP is drawn perpendicular to CP and AB is a diameter of the circle.

To prove that AB : AC = AC : AP (or, $AC^2 = AB.AP$). Join BC.

Proof : $\angle ACB$ being in a semi-circle is a right angle and $\angle P$ is also a right angle.

$\because$ PC is a tangent and CA is a chord through the point of contact, $\therefore$ $\angle ACP = \angle B$ in the alternate segment.

$\therefore$ $\triangle$ˢABC, APC are equiangular. $\therefore$ $\frac{AB}{AC}=\frac{AC}{AP}$.

Ex. 11. A, B, C and D are four points lying in order on a straight line. Find a point X on it such that XA : XB = XC : XD. [C. U. '42 ; E. B. S. B. '50]

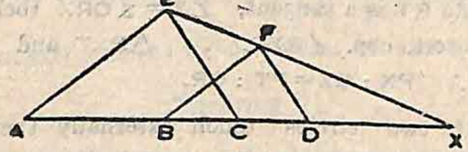


Fig. 38

Let A, B, C and D be four points on the same st. line. To find a point X on the st. line so that XA : XB = XC : XD.

Construction : On AC draw any $\triangle$AEC. From B and D draw st. lines parallel respectively to AE and CE and let them meet at F. Join EF and produce it to meet AD produced at X. Then X is the point required.

Proof : $\because$ BF ∥ AE, $\therefore$ $\frac{AX}{BX}=\frac{EX}{FX}$.

Again, $\because$ DF ∥ CE, $\therefore$ $\frac{CX}{DX}=\frac{EX}{FX}$. $\therefore$ $\frac{AX}{BX}=\frac{CX}{DX}$.

Ex. 12. A common tangent to two circles cuts the line joining their centres, internally or externally, in the ratio of their radii.

Let P, Q be the centres of the circles and let their common tangent TR cut PQ, the line of centres, internally (in the first fig.) or externally (in the second fig.) at X.

To prove that PX : QX = PT : QR. Join PT and QR.

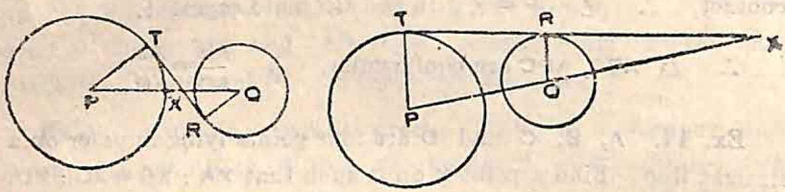


Fig. 39

Proof : As RT is a tangent, ∠T = ∠QRX (being rt. ∠s) and ∠PXT = vert. opp. ∠RXQ, ∴ △PXT and △QXR are equiangular. ∴ PX : QX = PT : QR.

Ex. 13. If two circles touch externally their common tangent is a mean proportional between their diameters.

Let two circles touch externally at T and let PQ be their common tangent.

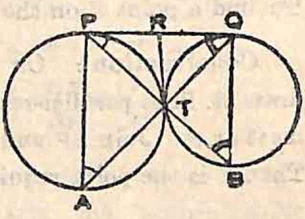


Fig. 40

To prove that PQ is the mean proportional between the diameters of the two circles.

Construction : Join PT and produce it to meet the circumference at B. Join QT and produce it to meet the circumference at A. Join PA, QB.

Proof : ∵ the circles touch at T, ∴ they have a common tangent at T. Let TR be the common tangent at T, meeting PQ at R.

Now, ∵ the tangents PR, RT are equal, ∴ ∠RTP = ∠RPT.

Again, ∵ RT = RQ (being tangents), ∴ ∠RTQ = ∠RQT.

∴ the whole ∠PTQ = ∠TPQ + ∠PQT = $\frac{1}{2}$ of 2 rt. angles

= 1 rt. angle.

$\therefore$ $\angle PTA = 1$ rt. angle, $\therefore$ it is an angle in the semi-circle. $\therefore$ PA is a diameter of the circle. Similarly, $\angle QTB$ is a right angle, and so QB is a diameter of the other circle.

$\because$ QP is a tangent and PT is a chord through the point of contact, $\therefore$ $\angle QPT = \angle A$ in the alt. segment.

Similarly, $\angle PQT = \angle B$ in the alt. segment.

Now in Δ^s PAQ and PBQ, $\angle AQP = \angle PBQ$ and $\angle PAQ = \angle BPQ$. $\therefore$ Their third angles are equal.

$\therefore$ the Δ^s are equiangular, $\therefore$ $\frac{AP}{PQ} = \frac{PQ}{BQ}$, that is, PQ is the mean proportional between the diameters AP and BQ.

Theorem 9

If two triangles have one angle of the one equal to one angle of the other and the sides about the equal angles proportional, the triangles are similar.

[C. U. '44, '46, '48, '50 ; D. B. '37, '43, '45 ; G. U. '48]

Let ABC and DEF be two triangles, in which $\angle A = \angle D$ and $AB : DE = AC : DF$.

To prove that the Δ^s ABC and DEF are similar.

Construction : From AB and AC cut off AX and AY equal to DE and DF respectively. Join XY.

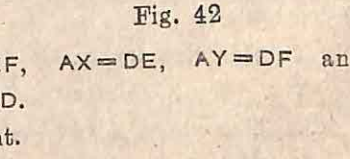


Fig. 42

Proof : In Δ^s AXY and DEF, $AX = DE$, $AY = DF$ and the included $\angle A =$ the included $\angle D$.

$\therefore$ The triangles are congruent.

$\therefore$ $\angle AXY = \angle DEF$, and $\angle AYX = \angle DFE$.

Again, $\because$ $\frac{AB}{DE} = \frac{AC}{DF}$, $\therefore$ $\frac{AB}{AX} = \frac{AC}{AY}$, $\therefore$ XY $\parallel$ BC.

$\therefore$ $\angle AXY =$ the corresponding $\angle ABC$, $\therefore$ $\angle E = \angle B$.

Similarly $\angle F = \angle C$. $\therefore$ the $\triangle^s$ ABC, DEF are equiangular and so their corresponding sides are proportional.

Hence the $\triangle^s$ ABC, DEF are similar.

Miscellaneous Solutions (5)

Ex. 1. In equiangular triangles the medians make equal angles with the corresponding sides. [cf. W. B. S. F. '52]

Let ABC and DEF be two triangles in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and let AP and DQ be the medians of the $\triangle ABC$ and $\triangle DEF$ respectively.

To prove that $\angle APB = \angle DQE$.

Proof : $\because$ $\triangle^s$ ABC, DEF are similar,

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{2BP}{2EQ} = \frac{BP}{EQ}.$$

Now, in $\triangle^s$ ABP and DEQ, $\angle B = \angle E$ and $AB : DE = BP : EQ$.

$\therefore$ the triangles are similar. $\therefore$ $\angle APB = \angle DQE$.

$\therefore$ The medians make equal angles with the corresponding sides. This is true for other corresponding medians also.

Ex. 2. In a triangle ABC, AD is drawn perpendicular to the base. If $BD : DA = DA : DC$, prove that the triangle ABC is a right-angled triangle.

[First give the statement]

Proof : In $\triangle^s$ ABD and ADC, we have $\angle ADB = \angle ADC$ (being right angles) and $BD : DA = DA : DC$ (hypothesis).

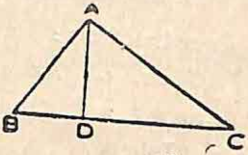


Fig. 42

$\therefore$ the triangles are similar.

$\therefore$ $\angle B = \angle DAC$ and $\angle C = \angle DAB$.

$\therefore$ The whole $\angle BAC = \angle DAB + \angle DAC = \angle B + \angle C$,

$\therefore$ $\angle BAC = \frac{1}{2}$ of 2 rt. angles = 1 rt. angle.

$\therefore$ $\triangle ABC$ is a right-angled triangle.

Ex. 3. The altitudes of two similar triangles are proportional to their corresponding sides.

[Draw the figure] Let $\triangle ABC$ and $\triangle DEF$ be similar and let $AM \perp BC$ and $DN \perp EF$. To prove that $\frac{AM}{DN}=\frac{BC}{EF}$.

Proof : In $\triangle ABM$ and $\triangle DEN$, $\angle B=\angle E$ (hyp.)

$\angle AMB=\angle DNE$ ($\because$ they are right angles),

$\therefore$ The triangles are equiangular, $\therefore$ $\frac{AM}{DN}=\frac{AB}{DE}=\frac{BC}{EF}$.

Ex. 4. OAB is a straight line, from A, B parallels AX and BY are drawn in the same sense so that OA : OB = AX : BY. Show that O, X and Y are collinear.

Let OAB be a st. line and let AX and BY be drawn parallel to each other so that OA : OB = AX : BY.

To prove that O, X and Y are collinear.

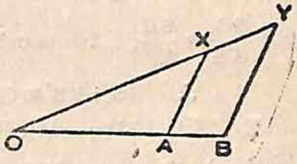


Fig. 43

Join OX and OY.

Proof : $\because$ $AX \parallel BY$, $\therefore$ $\angle OAX$ = corresponding $\angle OBY$.

Again, $\because$ OA : OB = AX : BY (hyp.), $\therefore$ $\triangle^s$ OAX, OBY are similar. $\therefore$ $\angle XOA=\angle YOB$, $\therefore$ OX and OY coincide.

$\therefore$ OX, OY are in the same st. line. $\therefore$ O, X and Y are collinear.

Ex. 5. The circum-radii of two similar triangles are proportional to their corresponding sides.

[**N. B.** The point of intersection of the perpendicular bisectors of any two sides of a triangle is the circum-centre of the triangle. So the distance between this point and any vertex of the triangle is the radius (*i.e.*, circum-radius) of the circum-circle of the triangle.]

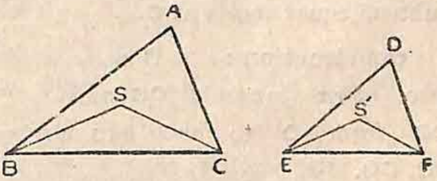


Fig. 44

Let S and S′ be the circum-centres of the two similar triangles ABC and DEF. Join BS, CS and ES′, FS′.

To prove that $\frac{BS}{ES'}=\frac{BC}{EF}$.

Proof : ∵ S is the centre of the circum-circle of the $\triangle ABC$, ∴ BS = CS (being radii). Again, the $\angle BSC$ at the centre and the $\angle A$ at the circumference are on the same arc, ∴ $\angle BSC = 2\angle A$.

Similarly, in $\triangle DEF$, $ES' = FS'$ and $\angle ES'F = 2\angle D$.

Since $\triangle^s ABC$, DEF are similar, ∴ $\angle A = \angle D$,

∴ $\angle S = \angle S'$. Now, in $\triangle^s BSC$, $ES'F$, since $\angle S = \angle S'$ and $\frac{SB}{S'E}=\frac{SC}{S'F}$, ∴ the ${}^s\triangle BSC$, $ES'F$ are similar.

∴ BS : ES′ = BC : EF.

Ex. 6. If the ratio of the tangents drawn from a given point O to two given circles be equal to the ratio of the radii of the corresponding circles, show that the two circles will subtend equal angles at O. [C. U. '12]

Let P and Q be the centres of two given circles and let OA and OB be drawn tangents from the given pt. O to the circles, so that

OA : OB = AP : BQ.

To prove that the circles subtend equal angles at O.

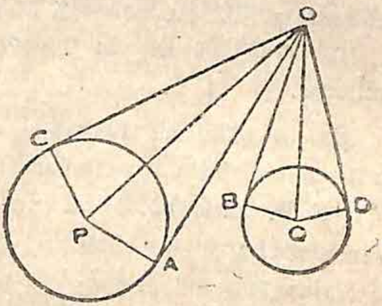


Fig. 45

Construction : Draw two other tangents OC and OD from O to the two circles. Join AP, PC, OP and BQ, DQ, OQ.

Proof : ∵ AP, BQ are radii drawn to the points of contact, ∴ each of the angles OAP and OBQ is a right angle. In $\triangle^s$ OAP and OBQ, $\because$ $\angle A = \angle B$ and OA : OB = AP : BQ (hyp.), $\therefore$ the two triangles are similar. $\therefore$ $\angle AOP = \angle BOQ$. Similarly it can be shown that $\triangle^s$ OPC, OQD are similar, $\therefore$ $\angle COP = \angle DOQ$.

$\therefore$ the whole $\angle AOC$ = the whole $\angle BOD$, that is, the given circles subtend equal angles at O.

Exercise 4

1. If one of the parallel sides of a trapezium is double of the other, show that its diagonals cut each other at the point of trisection.

2. Any two medians of a triangle cut one another at a point of trisection.

3. The bases BC and EF of two similar triangles ABC, DEF are divided in the same ratio at X and Y. Prove that AX : DY = BC : EF. [G. U. '48]

4. ABCD is a parallelogram; E and F are points in a st. line parallel to AB; AE and BF meet at P, and DE and CF meet at Q. Prove that PQ ∥ AD. [H. S. Tech. '60]

5. Find the locus of the middle point of a st. line parallel to the base of a triangle and terminated by the other two sides.

6. If two chords of a circle cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other. [C. U. '51; G. U. '52]

7. Prove that in two similar triangles the corresponding medians, altitudes, circum-radii and in-radii are proportional.

8. From a pt. P outside a circle, a tangent PT and a secant PAB are drawn, the latter cuts the circle at A and B. Prove that $PT^2 = PA.PB$.

9. The bisector of the $\angle A$ of the $\triangle ABC$ cuts BC in D and the circum-circle at P. Show that AB.AC = AP.AD.

10. In a $\triangle ABC$, AD is perpendicular to BC and $AB^2 = BC.BD$; show that the $\angle A$ is a right angle.

11. In $\triangle ABC$, AD is perpendicular from A to BC. If BD, AD and CD be in continued proportion, then the angle BAC is a right angle. [C. U. '44]

12. ABCD is a parallelogram ; a st. line drawn from A cuts BD, BC, DC in P, Q, R respectively. Prove that $AP^2 = PQ.PR$.

13. The bisector of the vertical angle A of the $\triangle ABC$ meets the base at D and the circumference of the circum-circle at E. If EC is joined, show that $AB.AC = AE.AD$.

14. In two circles, if any two parallel radii are drawn (one in each circle), the st. line joining their extremities cuts the line of centres in one or other of two fixed points.

[C. U. '17]

[**Hints** : (Draw the fig.) Let A, B be the centres of two given circles in which AP and BQ are two parallel radii. Let PQ cut AB internally at X and externally at Y. Now, $\triangle^s$ APX and BQX, and $\triangle^s$ APY, BQY are equiangular. $\therefore$ $AX : BX = AP : BQ$, and $AY : BY = AP : BQ$, $\because$ AP, BQ are fixed, and A and B are fixed points, being centres of two fixed circles, $\therefore$ X and Y are two fixed points.]

N. B. Either of the points of section, in which the st. line joining the centres of two given circles is divided internally and externally in the ratio of their radii, is called a **centrs of similitude** of the circles.

Theorem 10

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.

[C. U. '39 '43 '45 ; W. B. S. B. '53 ; D. B. '39]

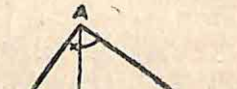


Fig. 46

Let ABC be a triangle having $\angle A$ a right angle and let AD be drawn perpendicular from A to the hypotenuse BC.

To prove that the $\triangle^s$ ABD and ADC are similar and each is similar to $\triangle ABC$.

Proof : In the $\triangle^s$ ABD and ABC,

the $\angle ADB = \angle BAC$ (being right angles), the $\angle B$ is common to both, and so the remaining angles BAD and ACB are equal.

$\therefore$ the triangles arc equiangular.

$\therefore$ their corresponding sides are proportional.

$\therefore$ They are similar.

Similarly, it can be proved that $\triangle^s$ ADC, ABC are equiangular and hence similar.

Again, $\because$ $\triangle ABD$ and $\triangle ADC$ are each equiangular to $\triangle ABC$,

$\therefore$ they are equiangular to one another, and hence similar.

Cor. : (1) The perpendicular is a mean proportional between the segments of the hypotenuse.

To prove that AD is the mean proportional between BD and DC.

Proof : In $\triangle^s$ ABD and ADC, $\angle ADB = \angle ADC$ (being rt. angles), $\angle ABD = \angle CAD$ and $\angle BAD = \angle ACD$,

$\therefore$ $\frac{BD}{AD} = \frac{AD}{CD}$, $\therefore$ AD is the mean proportional between BD and CD.

(2) $\because$ the $\triangle^s$ABD, ABC are similar, $\therefore$ $AB^2 = BC.BD$, *i.e.*, AB is the mean proportional between BC and BD.

(3) $\because$ $\triangle^s$ACD and ABC are similar, $\therefore$ $AC^2 = BC.CD$, *i.e.*, AC is the mean proportional between BC and CD.

Problem 4

To find the mean proportional between two given straight lines. [C. U. '46]

Let a and b be two given straight lines.

To find the mean proportional between a and b.

Construction : Take any straight line AX and from it cut off $AB = a$ and $BC = b$.

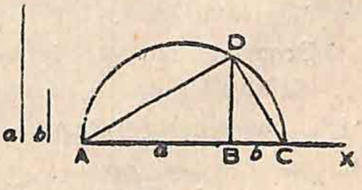


Fig. 47

On AC as diameter, draw a semi-circle.

Draw BD⊥AC and let BD cut the semi-circle at D. Then BD is the mean proportional between a and b.

Proof : Join AD and DC. The ∠ADC is a right angle, being in a semi-circle.

Thus ADC is a right-angled triangle in which DB is perpendicular to the hypotenuse from the right angle.

$\because$ $\triangle^s$ABD, DBC are similar, $\therefore$ $\frac{AB}{BD}=\frac{BD}{BC}$, *i.e.*, $\frac{a}{BD}=\frac{BD}{b}$.

Hence BD is the mean proportional between a and b.

[N.B. If AC, AB be the given st. lines and they are placed one upon the other as in fig. 47, then AD is the mean proportional between AC and AB. Again, CD is the mean proportional between CA and CB.]

Miscellaneous Solutions (6)

Ex. 1. The altitude AD of a △ABC, right-angled at A, cuts BC in D. Prove that BD : AD = AD : DC. [C. U. '48]

[Draw the fig. of Theorem 10, and prove that △ABD and △ADC are similar, $\therefore$ BD : AD = AD : DC.]

Ex. 2. N is the foot of the perpendicular from a point P of the circle APB upon a diameter AB. Prove that PB^2 = AB.NB. [C. U. '39]

Let P be a point on the circumference of the circle drawn on AB as diameter.

Let PN be perpendicular to AB.

To prove that PB^2 = AB.NB.

Join AP and BP.

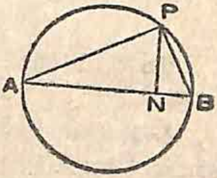


Fig. 48

Proof : $\because$ AB is a diameter, $\therefore$ ∠APB being in a semi-circle is a right angle. $\because$ PN⊥AB, $\therefore$ △ APB and △PNB are similar.

$\therefore$ $\frac{AB}{PB}=\frac{PB}{NB}$, $\therefore$ PB^2 = AB.NB.

Ex. 3. If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse and if the sides of the right-angled triangle are in continued proportion, the greater segment of the hypotenuse is equal to the smaller side of the triangle. [C. U. '39]

[Draw the fig. of Theorem 10] Let ABC be a triangle, right-angled at A and AD be perperdicular to the hypotenuse BC. Let AB, AC, BC be in continued proportion,

i.e., $AB : AC = AC : BC$.

If AB be the smallest side of the triangle and $CD > DB$, then prove that $CD = AB$.

Proof : $\triangle ACD$ and $\triangle ABC$ are similar, $\therefore \frac{CD}{AC} = \frac{AC}{BC}$.

But $\frac{AB}{AC} = \frac{AC}{BC}$ (hyp.), $\therefore \frac{CD}{AC} = \frac{AB}{AC}$. $\therefore CD = AB$.

Ex. 4. Find geometrically the value of $\sqrt{5}$.

[See the fig. of Problem 4] Take a st. line $AC = 5$ units of length and from it cut off $AB = 1$ unit of length.

Draw AD, the mean proportional between AB and AC.

Then $\frac{AB}{AD} = \frac{AD}{AC}$, or, $AD^2 = AB.AC = 5 \times 1$ or 5 units of area.

$\therefore AD = \sqrt{5}$ units of length. Then the length AD is the geometrical value of $\sqrt{5}$.

[N. B. $\sqrt{15} = \sqrt{5 \times 3}$, so the mean proportional between 5 and 3 units is the value of $\sqrt{15}$. Similarly, $\sqrt{34} = \sqrt{6{\cdot}8 \times 5}$]

Exercise 5

1. Find the mean proportional between 3 cms. and 4 cms.
2. Find geometrically the values of $\sqrt{35}$ and $\sqrt{26}$.

3. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the segments of the hypotenuse are in the duplicate ratio of the sides containing the right angle.

4. Two circles intersect at A and B ; and at A tangents are drawn, one to each circle, to meet the circumferences at P and Q. Show that $AB^2 = BP.BQ$.

5. In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$; prove that $\triangle ABC : \triangle DEF = BA.AC : ED.DF$. [C. U. '47]

6. Two circles of radii r, p respectively touch externally at A, and a common tangent touches them at P and Q. Prove that $PQ^2 = 4pr$.

7. If two triangles have one angle of the one equal to one angle of the other, prove that their areas are proportional to the rectangles contained by the sides that include the angle.

[W. B. S. F. '53]

Theorem 11

The ratio of the areas of two triangles of equal altitude is equal to the ratio of their bases.

Le ABC and DEF be two triangles on the bases BC and EF respectively, and let h be their altitude.

To prove that

$\triangle ABC : \triangle DEF = BC : EF$.

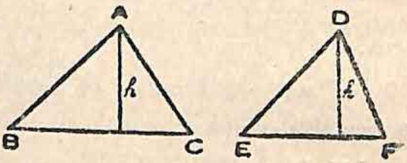


Fig. 46

Proof : $\because$ Area of a triangle $= \frac{1}{2}$ base $\times$ altitude,

$$\therefore \quad \triangle ABC = \tfrac{1}{2} BC.h \text{ and } \triangle DEF = \tfrac{1}{2} EF.h.$$

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} BC.h}{\frac{1}{2} EF.h} = \frac{BC}{EF}.$$

Theorem 12

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

[C.U. '45,'47,'48,'51 ; D.B. '39,'40,'45,'46,'48 ; G.U. '49,'51]

Or, Similar triangles are to one another in the duplicate ratio of their homologus sides.

Let ABC and DEF be two similar triangles in which $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and BC, EF are corresponding sides.

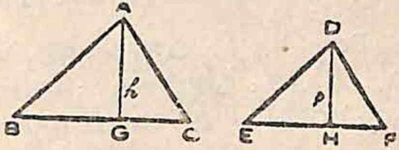


Fig. 50

To prove that $\triangle ABC : \triangle DEF = BC^2 : EF^2$.

Construction : Draw $AG \perp BC$ and $DH \perp EF$.

Let $AG = h$ units of length and $DH = p$ units of length.

Then h and p are the altitudes of the triangles.

Proof. $\because$ Area of a triangle $= \frac{1}{2}$ base $\times$ altitude,

$\therefore$ $\triangle ABC = \frac{1}{2} BC.h$ and $\triangle DEF = \frac{1}{2} EF.p$.

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC.h}{\frac{1}{2}EF.p} = \frac{BC}{EF}.\frac{h}{p}.$$

Now, in $\triangle ABG$ and $\triangle DEH$, $\angle B = \angle E$ and $\angle AGB = \angle DHE$ (being rt. angles), $\therefore$ The triangles are equiangular.

$$\therefore \quad \frac{h}{p} = \frac{AB}{DE} = \frac{BC}{EF} \quad (\because \ \triangle ABC \text{ and } \triangle DEF \text{ are similar.})$$

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF}.\frac{h}{p} = \frac{BC}{EF}.\frac{BC}{EF} = \frac{BC^2}{EF^2}.$$

[**N. B.** $a^2 : b^2$ is called the duplicate ratio of $a : b$. Homologus = corresponding.]

Corollary : Areas of two similar polygons are to one another as the squares on their corresponding sides. [See Appendix]

Miscellaneous Solutions (7)

Ex. 1. Show that every quadrilateral is divided by its diagonals into four triangles whose areas are proportional.

[D. B. '42]

Let the diagonals AC, BD of the quadrilateral ABCD intersect at O. To prove that $\triangle AOB : \triangle BOC = \triangle AOD : \triangle COD$.

Proof : $\because$ $\triangle AOB$ and $\triangle BOC$ are of the same altitude, $\therefore$ $\triangle AOB : \triangle BOC = AO : CO$. Similarly, $\triangle AOD : \triangle DOC = AO : CO$. $\therefore$ $\triangle AOB : \triangle BOC = \triangle AOD : \triangle COD$.

Ex. 2. Any triangle described on a diagonal of a square is double the similar triangle described on the side of the square.

[Draw the fig.] Let AEC and ABF be two similar triangles drawn on the diagonal AC and the side AB respectively of the square ABCD.

To prove that $\triangle AEC = 2\triangle ABF$.

Proof. $\because$ $\triangle AEC$ and $\triangle ABF$ are similar,

$$\therefore \frac{\triangle AEC}{\triangle ABF} = \frac{AC^2}{AB^2} = \frac{AB^2 + BC^2}{AB^2} \quad (\because \angle ABC \text{ is a rt. angle})$$

$$= \frac{2AB^2}{AB^2} = 2, \quad \therefore \triangle AEC = 2\triangle ABF.$$

Ex. 3. ABC is a triangle right-angled at A and AD is drawn perpendicular to BC. Show that $\triangle ABD : \triangle ACD = AB^2 : AC^2$.

[G. U. '49]

[Draw fig. no. 46] In $\triangle^s ABD, ACD$, the $\angle ADB = \angle ADC$ (being right angles), $\angle B = \angle CAD$ and $\angle BAD = \angle C$, so that AB and AC are two corresponding sides.

$\therefore$ The two triangles are similar.

$\therefore$ $\triangle ABD : \triangle ACD = AB^2 : AC^2$.

Ex. 4. Show that any given triangle is divided into 4 equal parts by the st. lines joining the middle points of its sides.

[C. U. '11, '40]

[Draw a fig.] Let ABC be a triangle in which D, E and F are the middle points of the sides AB, BC and AC respectively. Join DE, EF and FD. To prove that $\triangle^s$ ADF, BDE, ECF, DEF are equal in area.

Proof : $\because$ DF $\parallel$ BC, $\therefore$ $\angle$ADF = the corresponding $\angle$B, and $\angle$AFD = cor. $\angle$C. $\therefore$ $\triangle^s$ADF, ABC are equiangular and hence similar.

$$\therefore\ \frac{\triangle ADF}{\triangle ABC}=\frac{AD^2}{AB^2}=\frac{AD^2}{(2AD)^2}=\frac{AD^2}{4AD^2}=\frac{1}{4}.\ \therefore\ \triangle ADF=\tfrac{1}{4}\text{ of }\triangle ABC.$$

Similarly, it can be proved that each of the triangles DBE and ECF is equal to $\frac{1}{4}$ of $\triangle$ABC, hence the remaining $\triangle$DEF $=\frac{1}{4}$ of $\triangle$ABC.

$\therefore$ The $\triangle$ABC is divided into four equal parts.

Ex. 5. Equilateral triangles are described on the sides of a right-angled triangle. Prove that the area of the triangle on the hypotenuse is equal to the sum of the areas of the other two triangles.

Let the $\angle$A of the $\triangle$ABC be a rt. angle and let X, Y, Z be equilateral triangles on AB, AC, BC respectively. To prove that Z = X + Y.

Proof : $\because$ $\triangle^s$ X, Y, Z are equilateral, $\therefore$ each of their angles is 60°. $\therefore$ they are equiangular and hence similar triangles.

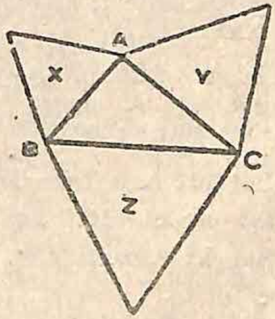


Fig. 51

$$\therefore\ \frac{X}{Z}=\frac{AB^2}{BC^2}\ \text{and}\ \frac{Y}{Z}=\frac{AC^2}{BC^2}.$$

$$\therefore\ \frac{X}{Z}+\frac{Y}{Z}=\frac{AB^2}{BC^2}+\frac{AC^2}{BC^2},$$

that is, $\frac{X+Y}{Z}=\frac{AB^2+AC^2}{BC^2}=\frac{BC^2}{BC^2}$ [$\because$ $\angle A=1$ rt. $\angle$] $=1$.

$\therefore \quad X+Y=Z.$

Ex. 6. Draw an equilateral triangle equal to the sum of two given equilateral triangles. [D. B. '46]

[**Hints** : Let ABC and DEF be the two given equilateral triangles. Take a st. line $PQ=BC$. Draw $PR \perp PQ$, making $PR=EF$. Join QR. The equilateral triangle on QR is the triangle required. For proof, see solution of Ex. 5. above.]

Ex. 7. Draw an equilateral triangle equal to the difference of two given equilateral triangles. [C. U. '45]

[**Hints** : Let ABC and DEF be two equilateral triangles, $\triangle ABC$ being greater than $\triangle DEF$. Take a st. line $KM=EF$ and draw $MP \perp KM$. With centre K and with radius BC, draw an arc of a circle, cutting MP at N. Then the equilateral $\triangle MNR$ drawn on MN is the triangle required.

Proof : $\because$ The three triangles here are equilateral,

$\therefore$ they are similar. $\therefore$ $\frac{\triangle ABC}{\triangle MNR}=\frac{BC^2}{MN^2}$ and $\frac{\triangle DEF}{\triangle MNR}=\frac{EF^2}{MN^2}$.

$$\therefore \quad \frac{\triangle ABC-\triangle DEF}{\triangle MNR}=\frac{BC^2-EF^2}{MN^2}=\frac{KN^2-KM^2}{MN^2}$$

$=\frac{MN^2}{MN^2}$ ($\because$ $\angle M$ is a rt. $\angle$) $=1$. $\therefore$ $\triangle MNR=\triangle ABC-\triangle DEF$.

Ex. 8. ABC is a triangle inscribed in a circle. If the tangent at A meets BC produced in D, prove that $CD : BD=AC^2 : AB^2$. [C. U. '28, '39, '51 ; G. U. '51]

Let ABC be a triangle inscribed in the circle ABC. Let AD be a tangent at A meeting BC preduced at D.

To prove that, $CD : BD=AC^2 : AB^2$.

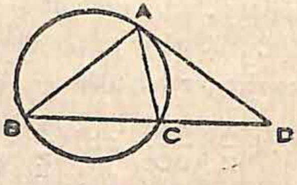


Fig. 52

Proof : In $\triangle$'s ACD and ABD, the $\angle D$ is common and $\angle DAC=\angle ABC$ in the alternate segment.

Then $\angle ACD = \angle BAD$. $\therefore$ The triangles are equiangular and hence similar. $\therefore$ $\triangle ACD : \triangle ADB = AC^2 : AB^2$.

Again, $\because$ $\triangle^s$ ACD and ABD are of the same altitude,

$\therefore$ $\triangle ACD : \triangle ABD = CD : BD$. $\therefore$ $CD : BD = AC^2 : AB^2$.

Ex. 9. Prove that the areas of similar triangles have the same ratio as the squares on corresponding medians. [D. B. '38]

[Draw the fig.] Let AG and DH be the medians of the similar $\triangle^s$ ABC, DEF respectively.

To prove that $\triangle ABC : \triangle DEF = AG^2 : DH^2$.

Proof : $\because$ $\triangle ABC$ and $\triangle DEF$ are similar,

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{2BG}{2EH} = \frac{BG}{EH}.$$

Now, in $\triangle^s$ ABG and DEH, the $\angle B = \angle E$ and $\frac{AB}{DE} = \frac{BG}{EH}$.

$\therefore$ The $\triangle^s$ are similar, $\therefore$ $\frac{AG}{DH} = \frac{AB}{DE}$.

$$\text{Now, } \frac{\triangle ABC}{\triangle DEF} = \frac{AB^2}{DE^2} = \frac{AG^2}{DH^2}.$$

Ex. 10. If two triangles are similar, their areas are proportional to the squares on their corresponding altitudes.

[**Hints :** Let h and p be the altitudes of the triangles.

$$\frac{h}{p} = \frac{BC}{EF}, \quad \therefore \quad \frac{h^2}{p^2} = \frac{BC^2}{EF^2} = \frac{\triangle ABC}{\triangle DEF}. \;]$$

Ex. 11. Prove that similar triangles are to one another as the squares on the radii of their in-circles. [C. U. '40]

Let ABC, DEF be two similar triangles. In $\triangle$ABC, the bisectors of the angles B and C meet at P and PR is drawn perpendicular to BC.

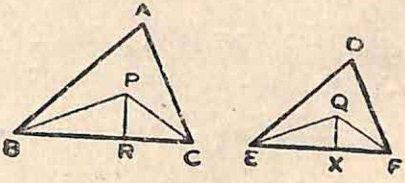


Fig. 53

Then P is the in-centre and PR the in-radius of the in-circle of the $\triangle ABC$. Similarly let Q be the in-centre and QX the in-radius of the in-circle of the $\triangle DEF$.

To prove that $\triangle ABC : \triangle DEF = PR^2 : QX^2$.

Proof : $\angle PBC = \frac{1}{2}\angle ABC$, and $\angle QEF = \frac{1}{2}\angle DEF$, but $\angle ABC = \angle DEF$ ($\because$ $\triangle^s$ ABC, DEF are similar),

$\therefore$ $\angle PBC = \angle QEF$. Similarly $\angle PCB = \angle QFE$.

$\therefore$ $\triangle^s$ PBC, QEF are equiangular, $\therefore$ $PB : QE = BC : EF$.

Again, in $\triangle^s BPR$, EQX, the $\angle PBR = \angle QEX$ and $\angle PRB = \angle QXE$ (being rt. $\angle s$), $\therefore$ The two triangles are equiangular, $\therefore$ $PR : QX = BP : QE$. $\therefore$ $PR : QX = BC : EF$.

Hence, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{BC^2}{EF^2} = \dfrac{PR^2}{QX^2}$,

Ex. 12. Prove that similar triangles are to one another as the squares on the radii of their circum-circles.

[C. U. '30, '36 ; D. B. '43]

[See Ex. 5 of Miscellaneous Solutions (4). Accordingly, first shew that, $\dfrac{SB}{S'E} = \dfrac{BC}{EF}$.

$\therefore$ $\triangle ABC : \triangle DEF = BC^2 : EF^2 = SB^2 : S'E^2$.]

Ex. 13. If DE is drawn parallel to the base BC of a triangle ABC and if $AD : DB = 3 : 2$, find the ratio of $\triangle ADE$: fig. $DBCE$. [Draw a fig.]

$\because$ $\dfrac{AD}{DB} = \dfrac{3}{2}$, $\therefore$ $\dfrac{DB}{AD} = \dfrac{2}{3}$, $\therefore$ $\dfrac{DB + AD}{AD} = \dfrac{2+3}{3}$, $\therefore$ $\dfrac{AB}{AD} = \dfrac{5}{3}$.

Now, $\because$ $DE \parallel BC$, $\therefore$ $\angle ADE = \angle B$, $\angle AED = \angle C$,

$\therefore$ $\triangle^s$ ABC and DEF are equiangular and hence similar.

$\therefore$ $\dfrac{\triangle ABC}{\triangle ADE} = \dfrac{AB^2}{AD^2} = \dfrac{5^2}{3^2} = \dfrac{25}{9}$; $\therefore$ $\dfrac{\triangle ABC - \triangle ADE}{\triangle ADE} = \dfrac{25-9}{9}$

$\therefore$ $\dfrac{\text{fig. } DBCE}{\triangle ADE} = \dfrac{16}{9}$. $\therefore$ $\triangle ADE$: fig. $DBCE = 9 : 16$.

Ex. 14. Bisect a triangle by a straight line drawn parallel to the base. [C. U. '16, '32, '50 ; D. B. '46]

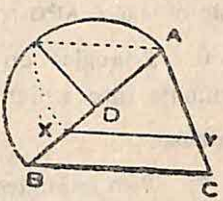


Fig. 54

Let ABC be the given triangle. To bisect it by a straight line drawn parallel to BC.

Construction : Bisect AB at D. From AB cut off AX equal to the mean proportional between AD and AB.

Through X, draw XY ∥ BC and let XY cut AC at Y. Then $\triangle$ABC is bisected by XY.

Proof : $\because$ AX is the mean proportional between AD and AB, $\therefore$ $AX^2 = AD.AB$. Again, $\because$ XY ∥ BC, $\therefore$ $\angle X = \angle B$, and $\angle Y = \angle C$. $\therefore$ $\triangle^s$ AXY, ABC are equiangular and hence similar.

$$\therefore \quad \frac{\triangle AXY}{\triangle ABC} = \frac{AX^2}{AB^2} = \frac{AD.AB}{AB^2} = \frac{AD}{AB} = \tfrac{1}{2} \; [\because \text{D is the mid pt. of AB.}]$$

Hence, $\triangle$ABC is bisected by XY.

Exercise 6

1. The area of the $\triangle$ABC is 19·6 sq. cm., and XY, drawn parallel to BC, cuts AB in the ratio 4 : 3. Find the area of the $\triangle$AXY. [Ans. = 6·4 sq. cm.]

2. Two equiangular triangles have areas in the ratio of 3 : 2 and an altitude of the greater is 5·2 cm. What is the corresponding altitude of the other ? [C. U. '35]

3. If PQ is drawn parallel to BC, the base of the $\triangle$ABC, and if $\triangle$APQ : fig. PBCQ = 4 : 5, show that AP : BP = 2 : 1.

4. Show how to draw a st. line XY parallel to BC, the base of a triangle ABC, so that the area of the $\triangle$AXY may be $\frac{9}{16}$ths of the $\triangle$ABC. [C. U. '26]

5. A trapezium ABCD has its sides AB, CD parallel and its diagonals intersect at P. If AB is double of CD, find the ratio of the $\triangle$APB to the $\triangle$PCD.

6. Triangles on the same base are to one another as the segments into which the join of their vertices is divided by the base.

7. Two triangles stand on equal bases and between the same parallels and a straight line is drawn parallel to their bases, show that it cuts off equal areas from the two triangles,

8. Prove that if similar triangles are drawn on the sides of a right-angled triangle, the area of the triangle described on the hypotenuse is equal to the sum of the areas of the other two triangles. [C. U. '45, '48]

9. The sides DA, CB of a cyclic quadrilateral ABCD are produced to meet in O. If $AB = \frac{1}{2}CD$, then the quadrilateral is three times the triangle OAB.

10. $\triangle$DEF is the pedal triangle of the $\triangle$ABC. Prove that $\triangle ABC : \triangle DBF = AB^2 : BD^2$. [W. B. S. F. '53]

11. Bisect a triangle by a straight line perpendicular to one of the sides.

TRIGONOMETRY

1. **'Trigonometry'** in its original sense signifies the measurement of triangles (*i.e.*, the relation that exists between the sides, angles and areas of trianges). Now it has a much wider scope and is a subject that deals with measurement of any angle.

2. Angles in Geometry and Trigonometry.

In Geometry we have supposed that angles are formed by the intersection of two straight lines and hence are confined in magnitude between 0° and 360°. The angles in geometry are always positive.

In Trigonometry the idea of an angle is much winder and an angle is supposed to be formed by a revolving straight line starting from an original position and revolving about one extremity.

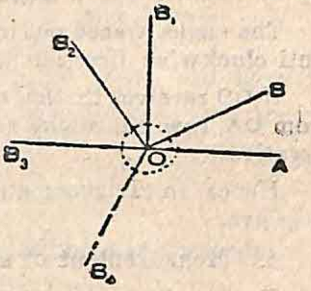


fig. 1

Suppose the st. line OB is revolving in the anti-clockwise direction about the point O, starting from its original position OA. In course of its revolution when it occupies the position OB it forms an acute angle AOB, when it comes to the position OB_1 perpendicular to OA it traces out a right angle AOB_1 (90°). Revolving further it comes to the position OB_2 and forms an obtuse angle AOB_2. When OB is thus opposite to and in the same st. line with OA, it will produce the straight angle (2 rt. angles) AOB_3 (180°). Thus revolving further it will form the reflex angle AOB_4. When OB makes a complete revolution it will coincide with OA and make an angle equal to four right angles (360°).

Now, if after making one complete revolution it again revolves in the same direction, it will evidently form an angle greater than four right angles. Thus OB may make any number of complete revolutions through the original position OA before taking up its final position. Hence we find that in Trigonometry angles may be of any magnitude and are not restricted as in Geometry.

The point O is called the origin, OA the initial position and OB is called the **generating line** or **radius vector**.

3. Positive and Negative angles.

The angles traced out by the radius vector rovolving in the **anti-clockwise** direction are considered to be positive.

If OB revolves in the clockwise direction about O starting from OA, then the angles traced out by it are considered to be negative.

Hence in Trigonometry the angles may be positive and negative.

4. Measurement of angles.

For measuring angles we must have some fixed unit. There are three different systems of units for measurement of angles, viz., the (1) **Sexagesimal unit**, (2) **Centesimal unit** and (3) **Circular unit.**

5. Sexagesimal (or English) system.

In this system the right angle, which is taken as the unit, is divided into 90 equal parts and each part is called a **degree.** A degree is divided into 60 equal parts and each part is called a sexagesimal **minute**, and a minute is again divided into 60 equal parts, each part being called a sexagesimal **second.**

Thus, $60''$ (sexagesimal seconds) $= 1'$ (minute)
$60'$ („ minutes) $= 1^\circ$ (degree)
90° (degrees) $= 1$ right angle.

6. Centesimal (or French) system :

In this system also the right angle is the unit and it is divided into 100 equal parts, each part being called a **grade.** A grade is divided into 100 equal parts, each part being called a centesimal **minute**, and a minute is again divided into 100 equal parts and each part is called a centesimal **second.**

Thus, 1 rt. angle $= 100^g$ (grades)

$1^g \quad = 100^{\backprime}$ (centesimal minutes)

$1^{\backprime} \quad = 100^{\backprime\backprime}$ (centesimal seconds)

[N. B. $1'$ indicates 1 sexagesimal minute, while $1^{\backprime}$ indicates 1 centesimal minute, and $1''$ denotes 1 sexagesimal second, while $1^{\backprime\backprime}$ denotes 1 centesimal second.]

7. The relation between the two systems of units.

$$90^\circ = 1 \text{ right angle} = 100^g, \quad \therefore \quad 1^\circ = \frac{10^g}{9}, \text{ and } 1^g = \frac{9^\circ}{10}.$$

Thus to pass from one system to the other, we have to express the given angle in terms of a right angle and then reduce it to the other system.

Examples 1

Ex. 1. Express 51036″ in degrees, minutes and seconds.

$$\begin{array}{r|l} 60 & 51036'' \;(\,850' \\ & \underline{480} \\ & 303 \\ & \underline{300} \\ & 36'' \end{array} \qquad \begin{array}{r|l} 60 & 850' \;(\,14^\circ \\ & \underline{60} \\ & 250 \\ & \underline{240} \\ & 10' \end{array} \quad \therefore\; 51036'' = 14^\circ 10' 36''.$$

Ex. 2. Express $10^\circ 36' 24''$ in seconds.

$$\begin{array}{r} 10^\circ 36' 24'' \\ \underline{60} \\ 600' \\ \underline{36'} \\ 636' \\ \underline{60} \\ 38160'' \\ \underline{24''} \\ 38184'' \end{array} \qquad \therefore\; 10^\circ 36' 24'' = 38184''.$$

Ex. 3. Express $24357^{\backprime\backprime}$ in grades, minutes and seconds.

$$\begin{array}{r|l} 100 & 24357^{\backprime\backprime} \\ \hline 100 & 243^{\backprime} \cdots\cdots 57^{\backprime\backprime} \\ \hline & 2^g \cdots\cdots 43^{\backprime} \end{array} \qquad \therefore\; 24357^{\backprime\backprime} = 2^g 43^{\backprime} 57^{\backprime\backprime}.$$

Ex. 4. Express $3^{g}9^{\backprime}74^{\backprime\backprime}$ in seconds.

$$\begin{array}{r}
3^{g}9^{\backprime}74^{\backprime\backprime} \\
100 \\ \hline
300^{\backprime} \\
9^{\backprime} \\ \hline
309^{\backprime} \\
100 \\ \hline
30900^{\backprime\backprime} \\
74^{\backprime\backprime} \\ \hline
30974^{\backprime\backprime}
\end{array}
\qquad \therefore\ 3^{g}9^{\backprime}74^{\backprime\backprime}=30974^{\backprime\backprime}.$$

Ex. 5. Express $56^{\circ}20'24''$ in grades, minutes and seconds.

$$56^{\circ}20'24''=56^{\circ}20'\frac{24}{60}=56^{\circ}20{\cdot}4'=56^{\circ}\frac{20{\cdot}4}{60}=56^{\circ}\frac{2{\cdot}04}{6}=56{\cdot}34^{\circ}$$

$$=\frac{56{\cdot}34}{90}\text{ rt. angle}={\cdot}626\text{ rt. angle}={\cdot}626\times 100\text{ grades.}$$

$$=62{\cdot}6^{g}=62^{g}60^{\backprime}.$$

Ex. 6. Express $30^{\circ}45'30{\cdot}6''$ in centesimal system.

$$30^{\circ}45'30{\cdot}6''=30^{\circ}+\left(45+\frac{30{\cdot}6}{60}\right)'=30^{\circ}+(45+{\cdot}51)'$$

$$=30^{\circ}+45{\cdot}51'=\left(30+\frac{45{\cdot}51}{60}\right)^{\circ}=(30{\cdot}7585)^{\circ}$$

$$=\frac{30{\cdot}7585}{90}\text{ rt. angle}$$

$$\begin{array}{rl}
={\cdot}34176\dot{1}\text{ rt. angle} & \\
\times 100 & \\ \hline
34{\cdot}176\dot{1}^{g} & \\
\times 100 & \\ \hline
17{\cdot}6\dot{1} & [\text{ from } {\cdot}176\dot{1}\times 100\ ] \\
\times 100 & \\ \hline
61{\cdot}\dot{1}^{\backprime\backprime} & [\text{ from } {\cdot}6\dot{1}\times 100\ ]
\end{array}$$

$$\therefore\quad 30^{\circ}45'30.6''=34^{g}17^{\backprime}61{\cdot}\dot{1}^{\backprime\backprime}.$$

Ex. 7. Express $14^{g}17^{\backprime}36^{\backprime\backprime}$ in right angles.

$$14^{g}17^{\backprime}36^{\backprime\backprime}=14^{g}17{\cdot}36^{\backprime}=14{\cdot}1736^{g}={\cdot}141736 \text{ rt. angle.}$$

[**N. B.** To reduce $36^{\backprime\backprime}$ into minutes we divide it by 100 and get ${\cdot}36^{\backprime}$. Thus $17^{\backprime}36^{\backprime\backprime}=17^{\backprime}{\cdot}36$. Dividing it by 100 we have ${\cdot}1736$ grades. Thus $14^{g}17^{\backprime}36^{\backprime\backprime}=14{\cdot}1736$ grades. Again dividing it by 100 we have ${\cdot}141736$ right angle.

Thus centesimal seconds may be reduced to centesimal minutes by placing the decimal point to the left of two digits from the right. This again can be reduced to grades by shifting the decimal point to the left of two more digits and similarly the grades can be reduced to right angles.]

Ex. 8. Express $40^{g}8^{\backprime}75^{\backprime\backprime}$ in sexagesimal system.

$$\begin{array}{rrl}
 & 40^{g}8^{\backprime}75^{\backprime\backprime}={\cdot}400875 \text{ rt. angle} & \\
 & \times 90 & \\ \hline
 & 36^{\circ}{\cdot}07875 & \\
 & \times 60 & \\ \hline
 & 4{\cdot}725' & [\text{ from } {\cdot}07875\times 60\ ] \\
 & \times 60 & \\ \hline
\therefore\ 40^{g}8^{\backprime}75^{\backprime\backprime}=36^{\circ}4'43{\cdot}5'' & 43{\cdot}5'' & [\text{ from } {\cdot}725\times 60\ ]
\end{array}$$

8. Circular or Radian Measure.

In this system a Radian is considered as the unit measure of angles. It is used in higher branches of Mathematics.

Radian : In any circle the angle subtended at the centre by an arc equal to its radius is called a **Radian** and it is written as 1^{c}.

Thus 4^{c} means an angle equal to 4 radians. A radian is a constant angle and is approximately equal to $57^{\circ}17'44{\cdot}8''$. Before further discussion we shall have the knowlege of the following theorem.

9. Theorem : "In all circles the ratio of the circumference to its diameter is always constant". [C. U. '48]

Or, "The circumferences of circles are to one another as their radii."

Or, "The length of the circumference of a circle bears a constant ratio to its diameter."

Let two concentric circles be drawn with centre O and any two radii R and r respectively, R being greater than r.

Let a regular polygon ABCD⋯ of n sides be inscribed in the first circle.

Join OA, OB, OC⋯cutting the smaller circle at n points $a, b, c, d\cdots$ If $ab, bc, cd,\cdots$ be joined, a regular polygon of n sides will be inscribed in the second circle.

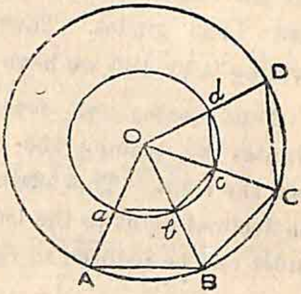


Fig. 2

Now, in $\triangle OAB$ and $\triangle Oab$, $\because$ $OA = OB$ and $Oa = Ob$,

$$\therefore \frac{OA}{Oa} = \frac{OB}{Ob}, \text{ and also } \because \angle AOB = \angle aOb,$$

$\therefore$ the two triangles are similar.

$$\therefore \frac{AB}{ab} = \frac{OA}{Oa} = \frac{R}{r}. \text{ Similarly } \frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \cdots = \frac{R}{r},$$

$$\therefore \frac{AB + BC + CD + \cdots}{ab + bc + cd + \cdots} = \frac{R}{r},$$

$$i.e., \frac{\text{the perimeter of the polygon ABCD}\ldots}{\text{the perimeter of the polygon } abcd\ldots} = \frac{R}{r}$$

$$= \frac{\text{radius of the circle ABCD}\cdots}{\text{radius of the circle } abcd\cdots}$$

The above relation holds good whatever may be the number of sides in the polygons. Evidently the lengths of AB, BC, CD......will decrease as n increases and gradually in the limit (when n is infinitely increased) the perimeter of

ABCD$\cdots$will coincide with the circumference of the circle and so the perimeter of the inner polygon will be the same as the circumference of the inner circle.

$$\therefore \frac{\text{circumference of } \odot \text{ ABCD}\cdots}{\text{circumference of } \odot\ abcd\cdots} = \frac{R}{r} = \frac{2R}{2r}$$

$$= \frac{\text{the diameter of } \odot \text{ ABCD}\cdots}{\text{the diameter of } \odot\ abcd\cdots} = \text{a constant quantity}$$

$$\therefore \frac{\text{circumference of } \odot \text{ ABCD}\cdots}{\text{diameter of } \odot \text{ ABCD}\cdots}$$

$$= \frac{\text{circumference of } \odot\ abcd\ldots}{\text{diameter of } \odot\ abcd\ldots} = \text{constant.}$$

The values of this constant ratio is denoted by the Greek letter π (pronounced as 'Pi').

This π is an incommensurable number approximately equal to $\frac{22}{7}$ and more accurately equal to 3·14159...

[Circumference of a circle $= 2\pi r$ (r being its radius) and area of a circle $= \pi r^2$.]

10. Theorem : A radian is a constant angle. [C. U. '46 ; G. U. 48]

[*Or*, The angle subtended at the centre of a circle by an arc which is equal in length to the radius is constant.]

Let APC be a circle whose centre is O and let AP be an arc = radius OA $= r$.

Then $\angle \text{AOP} = 1^c$ (one radian). It is required to prove that $\angle$AOP is constant.

Draw radius OB perpendicular to OA, then arc AB = $\frac{1}{4}$ of the circumference.

Proof : The angles at the centre of a circle are proportional to the arcs which subtend them.

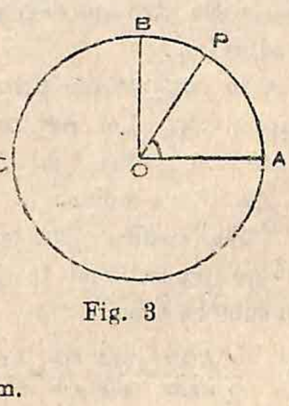


Fig. 3

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc AP}}{\text{arc AB}} = \frac{\text{radius OA}}{\frac{1}{4}\text{ circumference}} = \frac{r}{\frac{2\pi r}{4}} = \frac{2}{\pi}.$$

$$\therefore \quad \frac{\angle AOP}{1 \text{ rt. angle}} = \frac{2}{\pi}, \quad \therefore \quad \angle AOP = \frac{2}{\pi} \text{ right angle,}$$

$$\therefore \quad 1 \text{ radian} = \frac{2}{\pi} \text{ right angle} = \text{constant.}$$

Hence a radian is a constant angle.

[N. B. $1^c = \frac{2}{\pi}$ rt. angle $= \frac{180^\circ}{\pi} = 57^\circ 17' 44{\cdot}8''$ (App.) ; $\pi^c =$ 2 rt. angles $= 180^\circ$; $\frac{\pi^c}{2} = 1$ rt. angle ; $1^\circ = {\cdot}017453$ radian (App.)]

11. Theorem : The circular measure of an angle is equal to the ratio of the arc of any circle subtending that angle at its centre to the radius of the circle.

Or, Prove that the radian measure of any angle at the centre of a circle is expressed by the fraction $\frac{\text{subtending arc}}{\text{radius}}$.

[C. U. '40, '47]

Let MON be an angle whose circular measure is to be determined. Draw a circle with centre O and with any radius (r).

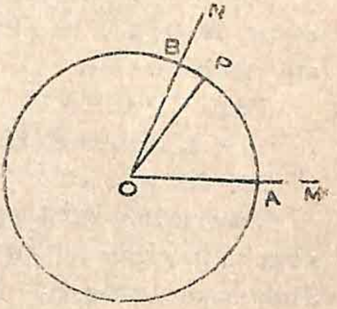


Fig. 4

Let it cut OM and ON at A and B respectively. Let AP be an arc equal to the radius (r). Join OP. Then $\angle AOP = 1$ radian.

$\because$ the angles at the centre of a circle are proportional to the arcs which subtend them,

$$\therefore \quad \frac{\angle AOB}{\angle AOP} = \frac{\text{arc AB}}{\text{arc AP}} = \frac{\text{arc AB}}{r}.$$

$$\therefore \quad \angle \text{MON} = \angle \text{AOB} = \frac{\text{arc AB}}{r} \times \angle \text{AOP}$$

$$= \frac{\text{arc AB}}{r} \times 1 \text{ radian} = \frac{\text{arc AB}}{r} \text{ radian.}$$

[**N. B.** If arc AB $= s$ and $\angle$AOB $= \theta$ radian, *i.e.*, θ^c, then $\theta = \frac{s}{r}$, *i.e.*, the **circular measure of an angle = arc ÷ radius.**]

12. Conversion from one system to another.

We have $90^\circ = 1$ rt. angle ; $100^g = 1$ rt. angle ; $\frac{\pi^c}{2} = 1$ rt. angle.

(1) $\because \quad 90^\circ = 1$ rt. angle ;

$$\therefore \quad 1^\circ = \tfrac{1}{90} \text{ rt. angle} = \frac{\pi}{180} \text{ radian} = \frac{\pi^c}{180},$$

$\because \quad 90^\circ = 100^g$; $\quad \therefore \quad 1^\circ = \frac{10}{9}$ grades.

(2) $\therefore \quad 100^g = 1$ rt. angle ; $\quad \therefore \quad 1^g = \frac{1}{100}$ rt. angle

$$= \frac{\pi}{200} \text{ radian} = \frac{\pi^c}{200}.$$

$\because \quad 100^g = 90^\circ$, $\quad \therefore \quad 1^g = \frac{9}{10}$ degree.

(3) $\because \pi^c = 2$ rt. angles $= 180^\circ$, $\therefore 1^c = \frac{180}{\pi}$ degrees $= \frac{180}{\pi}$.

$\because \quad \pi^c = 2$ rt. angles $= 200^g$, $\quad \therefore \quad 1^c = \frac{200^g}{\pi}$.

Suppose the measurement of an angle is D° in the sexagasimal measure, G^g in centesimal measure and R^c in circular measure:

Now, $\quad \because \quad 90^\circ = 1$ rt. angle, $\quad \therefore \quad D^\circ = \frac{D}{90}$ rt. angle.

Again, $\quad \because \quad 100^g = 1$ rt. angle, $\quad \therefore \quad G^g = \frac{G}{100}$ rt. angle.

Again, $\quad \because \quad \frac{\pi^c}{2} = 1$ rt. angle, $\quad \therefore \quad 1^c = \frac{2}{\pi}$ rt. angle.

$$\therefore \quad R^c = \frac{2R}{\pi} \text{ rt. angle.}$$

$$\therefore \quad \frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}, \qquad \therefore \quad \frac{D}{180} = \frac{G}{200} = \frac{R}{\pi}.$$

Examples 2

Ex. 1. Express 20°, 36°, 40^g and $6^g25^{\backprime}$ in radians.

(i) $\because\ 180^\circ=\pi^c$, $\therefore\ 1^\circ=\dfrac{\pi^c}{180}$, $\therefore\ 20^\circ=\dfrac{\pi^c\times 20}{180}=\dfrac{\pi^c}{9}$.

(ii) $\because\ 1^\circ=\dfrac{\pi^c}{180}$, $\therefore\ 36^\circ=\dfrac{\pi^c\times 36}{180}=\dfrac{\pi^c}{5}$.

(iii) $\because\ 200^g=\pi^c$, $\therefore\ 1^g=\dfrac{\pi^c}{200}$, $\therefore\ 40^g=\dfrac{\pi^c\times 40}{200}=\dfrac{\pi^c}{5}$.

(iv) $\because 1^g=\dfrac{\pi}{200}$ radian, $\therefore\ 6^g25^{\backprime}=6{\cdot}25^g=\dfrac{25^g}{4}=\dfrac{\pi^c}{200}\times\dfrac{25}{4}=\dfrac{\pi^c}{32}$.

Ex. 2. Express (*a*) $\frac{5}{12}\pi^c$ and (*b*) $\frac{9}{8}\pi^c$ in sexagesimal system.

(*a*) $\because\ 1^c=\dfrac{180^\circ}{\pi}$. $\therefore\ \dfrac{5}{12}\pi^c=\dfrac{180^\circ}{\pi}\times\dfrac{5\pi}{12}=75^\circ$.

(*b*) $\because\ 1^c=\dfrac{180^\circ}{\pi}$, $\therefore\ \dfrac{9\pi^c}{8}=\dfrac{180^\circ}{\pi}\times\dfrac{9\pi}{8}=\dfrac{45\times 9}{2}\text{ degrees}=\dfrac{405^\circ}{2}$

$=2$ rt. angles $22^\circ 30'$.

Ex. 3. Express $\dfrac{\pi^c}{12}$ in centesimal system.

$\because\ 1^c=\dfrac{200^g}{\pi}$, $\therefore\ \dfrac{\pi^c}{12}=\dfrac{200^g}{\pi}\times\dfrac{\pi}{12}=\dfrac{50^g}{3}=16{\cdot}\dot{6}^g=16^g66^{\backprime}66{\cdot}\dot{6}^{\backprime\backprime}$.

Ex. 4. Express in the three systems of angular measurement the magnitude of the interior angle of a regular pentagon.

$\because$ The interior angles of a polygon $+\,4$ rt. angles $=$ twice as many rt. angles as the figure has sides,

$\therefore$ the sum of the 5 equal interior angles of a regular pentagon $+\,4$ rt. angles $=10$ rt. angles,

$\therefore$ the sum of the 5 equal interior angles $=6$ rt. angles,

$\therefore$ each interior angle $=\frac{6}{5}$ rt. angles.

But 1 rt. angle $=90^\circ=100^g=\frac{\pi^c}{2}$,

$\therefore$ Each interior angle $=\frac{6}{5}\times 90^\circ=108^\circ$

$$\text{or,}\quad =\frac{6}{5}\times 100^g=120^g$$

$$\text{or,}\quad =\frac{6}{5}\times\frac{\pi^c}{2}=\frac{3\pi}{5}\text{ radians.}$$

Ex. 5. The difference between the two acute angles of a right-angled triangle is $\frac{\pi}{6}$ radian; express the angles in degrees.

Let $\angle A$ and $\angle B$ be two acute angles in the right-angled $\triangle ABC$, then $\angle A+\angle B=90^\circ\cdots(1)$,

and $\angle A-\angle B=\frac{\pi}{6}$ radian (hyp.) $=\frac{\pi}{6}\times\frac{180^\circ}{\pi}=30^\circ\cdots(2)$

Adding (1) and (2) we have $2\angle A=120^\circ$, $\therefore$ $\angle A=60^\circ$.

$$\therefore\quad \angle B=90^\circ-60^\circ=30^\circ.$$

$\therefore$ the angles are 60°, 30°, 90°.

Ex. 6. The angles of a triangle are in A.P., and the greatest is double the least; express the angles in radians.

Let the least angle be x degrees, so that the greatest angle is $2x$ degrees.

$\because$ the three angles are in A. P.,

$\therefore$ the third angle $=\frac{1}{2}(x+2x)$ or $\frac{3x}{2}$ degrees.

$\therefore$ $x+\frac{3x}{2}+2x=180^\circ$, or, $\frac{9x}{2}=180^\circ=\pi$ radians

$\therefore$ $x=\frac{2\pi^c}{9}$. $\therefore$ the angles are $\frac{2\pi}{9}$, $\frac{\pi}{3}$ and $\frac{4\pi}{9}$ radians respectively.

Miscellaneous Examples 3

Ex. 1. Two angles are in the ratio of 7 : 4 and their difference is 30°. Find the angles.

Let the angles be $7x^\circ$ and $4x^\circ$.

Then $7x^\circ - 4x^\circ = 30^\circ$, $\therefore$ $3x = 30$, $\therefore$ $x = 10$,

$\therefore$ one angle $= 7x^\circ = 70^\circ$ and the other $= 4x^\circ = 40^\circ$.

Ex. 2. The sum of two angles is 80 grades and their difference is 18°. Find the angles in degrees as well as in grades.

80 grades $= 80 \times \frac{9}{10}$ degrees $= 72^\circ$. Let A and B be the angles.

$$\begin{array}{l} \text{Then, } A + B = 72^\circ \\ \underline{\text{and } \ A - B = 18^\circ} \\ \therefore \ 2A \quad\; = 90^\circ, \quad \therefore \ A = 45^\circ. \quad \text{Then } B = 72^\circ - 45^\circ = 27^\circ. \end{array}$$

Again, $45^\circ = 45 \times \frac{10}{9}$ grades $= 50^g$, and $27^\circ = 27 \times \frac{10}{9}$ grades $= 30^g$.

Hence the two angles are 45° and 27°, or 50^g and 30^g.

Ex. 3. The sum of two angles is 135° and their difference is 100^g. Find the angles in radians. [G. U. '50]

Let the two angles be x and y degrees.

Now, $100^g = 90^\circ$, $\therefore$ from the given conditions

$$\begin{array}{l} x + y = 135^\circ \\ \underline{\text{and } x - y = \ 90^\circ} \\ \therefore \ 2x \quad = 225^\circ \end{array}$$

$$\therefore \ x = \frac{225^\circ}{2} = 112\tfrac{1}{2}^\circ \text{ and } y = 135^\circ - 112\tfrac{1}{2}^\circ = 22\tfrac{1}{2}^\circ$$

$$\therefore \ \text{one angle} = \frac{225^\circ}{2} = \frac{225}{2} \times \frac{\pi}{180} \text{ rdians} = \frac{5\pi}{8} \text{ radians,}$$

$$\text{and the other angle} = \frac{45^\circ}{2} = \frac{45}{2} \times \frac{\pi}{180} \text{ radian} = \frac{\pi}{8} \text{ radian.}$$

Ex. 4. Express in degrees the angle whose circular measure is $1{\cdot}309$. Assume $\pi = 3{\cdot}1416$. [C. U. '48]

$$\because \quad \pi^c = 180^\circ, \qquad \therefore \quad 1^c = \frac{180^\circ}{\pi},$$

$$\therefore \quad 1{\cdot}309 \text{ radians} = \frac{180 \times 1{\cdot}309}{3{\cdot}1416} \text{ degrees} = 75^\circ.$$

Ex. 5. Two angles of a triangle are $55^\circ 12' 36''$ and $64^\circ 47' 24''$. Find the third angle in centesimal measure.

[C. U. '50]

Sum of the given angles $= 55^\circ 12' 36'' + 64^\circ 47' 24'' = 120^\circ$,

but the sum of the three angles of a triagle $= 180^\circ$,

$\therefore$ the reqd. third angle $= 180^\circ - 120^\circ = 60^\circ = 60 \times \frac{10}{9}$ grades $= \frac{200}{3}$ grades $= 66{\cdot}\dot{6}$ grades $= 66^g\, 6\dot{6}^{`}\, 66{\cdot}\dot{6}^{``}$.

Ex. 6. Express in degrees and minutes and also in grades all the angles of an isosceles triangle in which each of the angles at the base is twelve times the vertical angle.

[C. U. '46]

Let the vertical angle be x°, then each base angle $= 12x^\circ$.

$\therefore \quad x^\circ + 12x^\circ + 12x^\circ = 180^\circ$, or, $25x^\circ = 180^\circ$,

$\therefore \quad x = 7\frac{1}{5}$ degrees $= 7^\circ 12'$. $\therefore$ the vertical angle $= 7^\circ 12'$,

and each base angle $= 7^\circ 12' \times 12 = 86^\circ 24'$.

Again, the vertical angle $= \frac{36}{5} \times \frac{10}{9}$ grades $= 8^g$

and each base angle $= 8^g \times 12 = 96^g$.

Ex. 7. Taking $\frac{1}{\pi} = {\cdot}31831$, show that a radian contains 206265 seconds approximately. [G. U. '48]

$$\because \quad \pi^c = 180^\circ, \quad \therefore \quad 1^c = 180^\circ \times \frac{1}{\pi} = 180^\circ \times {\cdot}31831$$
$$= 180 \times {\cdot}31831 \times 60 \times 60 \text{ seconds}$$
$$= 206264{\cdot}88 \text{ sec.} = 206265 \text{ sec.(app.)}.$$

Ex. 8. The sum of two angles is 114°. If the number of grades in one is equal to the number of degrees in the other, find the circular measure of the angles. [C. U. '43]

Let one angle be x°, then the other angle $= x^g = \dfrac{9x^\circ}{10}$.

Now, $x^\circ + \dfrac{9x^\circ}{10} = 114^\circ$, or, $\dfrac{19x}{10} = 114$, $\therefore \; x = \dfrac{114 \times 10}{19} = 60$.

$\therefore$ one angle $= 60^\circ = 60 \times \dfrac{\pi^c}{180} = \dfrac{\pi^c}{3}$,

and the other angle $= 60^g = 60 \times \dfrac{\pi^c}{200} = \dfrac{3\pi^c}{10}$.

Ex. 9. The angles of a triangle are in the ratio of $2 : 5 : 3$. Find the circular measure of the greatest angle. [C. U. '42]

Let ABC be the triangle, then $\angle A + \angle B + \angle C = 180^\circ$ and $\angle A : \angle B : \angle C = 2 : 5 : 3$.

$\therefore$ the greatest $\angle B = \dfrac{5}{2+5+3} \times$ sum of the angles

$$= \tfrac{5}{10} \times 180^\circ = 90^\circ = \frac{\pi^c}{2}.$$

Ex. 10. One angle of a triangle is 60° and the second is $\dfrac{\pi}{4}$ radian. Express the third angle in centesimal measure.

[C. U. '47]

Here, the first angle $= 60^\circ$, the second angle $= \dfrac{\pi^c}{4} = \dfrac{180^\circ}{4} = 45^\circ$;

but the sum of the three angles of a triangle $= 180^\circ$,

$\therefore$ the third angle $= 180^\circ - (60^\circ + 45^\circ) = 75^\circ = 75 \times \frac{10}{9}$ grades

$$= 83\tfrac{1}{3} \text{ grades} = 83^g 33' 33''\!\cdot\!\dot{3}.$$

Ex. 11. The angles of a triangle are in arithmetical progression. If the number of degrees in the greatest angle be same as the number of grades in the least one, find the angles in degrees. [C. U. '14]

Here the angles are in A. P.

Let them be $(a-d)$, a, $(a+d)$ degrees respectively.

$\because$ the sum of the 3 angles of a triangle$=180^\circ$,

$\therefore$ $(a-d)+a+(a+d)=180^\circ \cdots\cdots(1)$

Again, here the least angle is $(a-d)$ degrees and the greatest one is $(a+d)$ degrees.

Now, $(a-d)$ degrees$=\frac{10}{9}(a-d)$ grades.

$\therefore$ From the given condition $a+d=\frac{10}{9}(a-d)\cdots\cdots(2)$

Now from (1) we have $3a=180^\circ$, $\therefore$ $a=60^\circ$.

From (2) we have $60+d=\frac{10}{9}(60-d)$,

or, $540+9d=600-10d$, or, $19d=60$, $\therefore$ $d=\frac{60}{19}$.

Hence the angles are $(60-\frac{60}{19})$, 60 and $(60+\frac{60}{19})$ degrees respectively, *i.e.*, $56\frac{16}{19}$. 60 and $63\frac{3}{19}$ degrees.

Ex. 12. The radius of a circle is 6″, find the angle subtended at the centre by an arc 9″ in length.

We know arc $(s)=r\theta$*, here arc $(s)=9''$, $r=6''$. Let θ be the circular measure of the angle at the centre, $\therefore$ $9=6\theta$,

$$\therefore\quad \theta=\tfrac{9}{6} \text{ or } \tfrac{3}{2} \text{ radians}=\frac{3}{2}\times\frac{180}{\pi}\text{ degrees}=\frac{3\times180\times7}{2\times22}\text{ degrees}$$
$$=85\tfrac{10}{11}\text{ degrees.}$$

[***N. B.** In the formula $s=r\theta$, θ must always be in circular measure.]

Ex. 13. Find the length of an arc which subtends one minute at the centre of the earth, supposed to be a sphere of diameter 7920 miles. $(\pi=\frac{22}{7})$. [C. U. '48]

Arc $(s)=r\theta$. Here $r=$radius$=\frac{7920}{2}$ miles,

and $\theta=$angle at the centre$=1'=\frac{1}{60}$ degree$=\frac{1}{60}\times\frac{\pi}{180}$ radian.

$$\therefore\ \text{the length of the arc}=r\theta=\tfrac{7920}{2}\times\tfrac{1}{60}\times\tfrac{22}{7\times180}\text{ miles}$$
$$=\tfrac{121}{105}\text{ miles}=1\tfrac{16}{105}\text{ miles.}$$

Ex. 14. An arc of **17** yds. **1** ft. 3 in. subtends at the centre of a circle an angle of 1·9 radians. Find the radius of the circle in inches. [C. U. '49]

Arc $(s)=r\theta$; here arc $=17$ yds. 1 ft. 3 in. $=627$ inches,

and $\theta=$ angle at the centre $=1·9$ radians. $\therefore$ 627 in. $=r\times 1·9$,

$\therefore\ r=\frac{627}{1·9}$ in. $=\frac{6270}{19}$ in. $=330$ in. $\therefore$ the radius $=330$ inches.

Ex. 15. If the diameter of the earth is 8000 miles, find the measure of a Nautical mile. $[\pi=3·1416]$

The length of the arc of the longitude that subtends an angle of 1 minute at the centre of the earth is called a Nautical mile.

We know arc $=r\theta$; here $r=\frac{8000}{2}$ miles $=4000$ miles,

and $\theta=1'=\frac{1}{60}$ degree $=\frac{\pi}{60\times 180}$ radian

$\therefore$ a nautical mile $=r\theta=4000\times\frac{\pi}{60\times 180}$ miles

$$=\frac{4000\times 3·1416}{60\times 180}\text{ mi.}=\frac{10·472}{9}\text{ mi.}$$

$$=1·16 \text{ miles (app.).}$$

Ex. 16. A horse running along a circular track of radius 27 ft. passes over in 3 seconds an arc which subtends 70° at the centre. Find the distance the horse travels in half a minute. $(\pi=\frac{22}{7})$. [C. U. '51]

Arc $=r\theta$; here $r=27$ ft., $\theta=70^\circ=\frac{70\times\pi}{180}$ radians

$=\frac{70}{180}\times\frac{22}{7}$ radians $=\frac{11}{9}$ radians

$\therefore$ arc $=r\theta=27\times\frac{11}{9}$ ft. $=33$ ft.

$\therefore$ In 3 seconds the horse goes 33 ft.

$\therefore$ In $\frac{1}{2}$ minute or 30 seconds the horse goes 330 ft.

Ex. 17. A man running on a circular track at the rate of 10 miles an hour traverses an arc which subtends 56° at the centre in 36 seconds. Find the diameter of the circle. Given $\pi = \frac{22}{7}$. [C. U. '46]

In 1 hour or 3600 secs. he runs 10 miles or 17600 yds.

$\therefore$ in 36 ,, ,, 176 yds.

$\therefore$ the length of the arc $= 176$ yds.

The angle subtended at the centre

$$= 56° = \frac{\pi^c}{180} \times 56 = \frac{22 \times 56}{7 \times 180} \text{ radian} = \frac{44}{45} \text{ radian.}$$

Now, $\because$ arc $= r\theta$, $\therefore$ $176 = r \times \frac{44}{45}$, $\therefore$ $r = \frac{176 \times 45}{44} = 180$.

$\therefore$ the required diameter $= 2r = 180$ yds. $\times 2 = 360$ yds.

Ex. 18. If a_1, a_2, a_3 be the circular measures of the angles subtended by the arcs of lengths l_1, l_2, l_3 at the centres of circles whose radii are r_1, r_2, r_3 respectively, show that the angle subtended at the centre by the arc of length $l_1 + l_2 + l_3$ of a circle whose radius is $\frac{1}{n}(a_1r_1 + a_2r_2 + a_3r_3)$ will be n radians.

[C. U. '40]

$\because$ arc $= r\theta$, $\therefore$ $l_1 = r_1a_1$, $l_2 = r_2a_2$, $l_3 = r_3a_3$,

$\therefore$ $l_1 + l_2 + l_3 = a_1r_1 + a_2r_2 + a_3r_3$.

Now, if $l_1 + l_2 + l_3$ be the length of the arc and $\frac{1}{n}(a_1r_1 + a_2r_2 + a_3r_3)$ be the radius, then the angle subtended at the centre by this arc $= \frac{\text{arc}}{\text{radius}}$ $(\because r\theta = s]$

$$= \frac{l_1 + l_2 + l_3}{\frac{1}{n}(a_1r_1 + a_2r_2 + a_3r_3)} \text{ radians}$$

$$= \frac{(a_1r_1 + a_2r_2 + a_3r_3)}{\frac{1}{n}(a_1r_1 + a_2r_2 + a_3r_3)} \text{ rad.} = \frac{1}{\frac{1}{n}} \text{ rad.} = n \text{ radians.}$$

Ex. 19. If D and C be respectively the number of degrees and the number of radians in an angle, show that $\frac{D}{180}=\frac{C}{\pi}$.

[C. U. '47 ; D. B. '50]

Here the measure of the angle is D degrees or C radians.

∴ D degrees = C radians. ∴ D degrees $=\frac{D}{90}$ right angles and C radians $=\frac{2C}{\pi}$ right angles, ∴ $\frac{D}{90}=\frac{2C}{\pi}$, ∵ $\frac{D}{180}=\frac{C}{\pi}$.

Ex. 20. The number of degrees, grades and radians in an angle are respectively x, y and z. Show that $\frac{x}{90}=\frac{y}{100}=\frac{2z}{\pi}$.

[C. U. '41, 45 ; G. U. '51]

∵ the same angle measures x^{o}, y^{g} or z^{c}, ∴ $x^{o}=y^{g}=z^{c}$.

Now, $x^{o}=\frac{x}{90}$rt. angle, $y^{g}=\frac{y}{100}$ rt. angle and $z^{c}=\frac{2z}{\pi}$rt. angle

$$\therefore \quad \frac{x}{90}=\frac{y}{100}=\frac{2z}{\pi}.$$

Ex. 21. Find the area of a circle whose radius is **1** ft. ($\pi=3{\cdot}1416$).

The area of the circle $=\pi r^2=3{\cdot}1416\times({\cdot}1)^2$ sq. ft.

$=3{\cdot}1416\times{\cdot}01$ sq. ft. $={\cdot}031416$ sq. ft.

Ex. 22. A mill sail, 28 feet long, makes 10 revolutions per minute. What distance does its end traverse in an hour? ($\pi=\frac{22}{7}$).

Here radius $r=28$ ft., ∴ circumference $=2\pi r=2\times\frac{22}{7}\times28$ ft.

$=176$ ft.

∴ the sail traverses 176 ft. in 1 revolution,

∴ in 1 minute it traverses 176 ft. × 10 or 1760 ft.

∴ in 1 hour it traverses $\frac{1760\times60}{3\times1760}$ mi. or 20 miles.

Ex. 23. The large hand of a clock is 2 ft. 4 inches long; how many inches does its extremity move in 20 minutes? ($\pi = 3{\cdot}1416$). [C. U. '48]

The circumference of the dial of the clock $= 2\pi r$
$= 2 \times 3{\cdot}1416 \times 2$ ft. 4 in. $= 2 \times 3{\cdot}1416 \times 28$ inches.

$\because$ The minute-hand moves once round the dial in 60 minutes,

$\therefore$ It moves $\frac{1}{3}$ of the circumference of the clock in 20 minutes.

$\therefore$ The reqd. distance $= \dfrac{2 \times 3{\cdot}1416 \times 28}{3}$ in. $= 58{\cdot}6432$ inches.

Ex. 24. Find the times between 4 and 5 o'clock when the angle between the minute-hand and the hour-hand is (i) 72° and (ii) $\frac{8\pi}{15}$ radians.

The whole circumference of a clock subtends 360° at its centre. Again, the circumference of the clock is equal to 50 minute divisions.

(1) Now, 360° = angle at the centre subtended by an arc equal to 60 min. divisions,

$\therefore$ 72° = angle at the centre subtended by an arc equal to 12 min. divisions.

$\therefore$ The distance between the two hands will be 12 minute divisions.

Now, at 4 o'clock the minute-hand was 20 minute divisions behind the hour-hand. So, if the minute-hand gains *either* $(20-12)$, *i. e.*, 8 minute divisions *or* gains $(20+12)$, *i.e.*, 32 min. divisions over the hour-hand, then the angle between the two hands will be 72°.

The minute-hand gains 55 min. divisions in 60 minutes,

$\therefore$,, ,, ,, 8 ,, ,, $\dfrac{60 \times 8}{55}$ or $\dfrac{96}{11}$ or $8\dfrac{10}{11}$ minutes.

Again ,, ,, ,, 32 ,, ,, $\dfrac{60 \times 32}{55}$ or $34\dfrac{10}{11}$ minutes.

$\therefore$ The angle between the two hands will be 72° at $8\frac{8}{11}$ minutes past 4 and at $34\frac{10}{11}$ minutes past 4 o'clock.

(ii) $\frac{8\pi}{15}$ radians $=\frac{8\pi}{15}\times\frac{180^\circ}{\pi}=96^\circ$.

If the angle be 360°, the arc measures 60 min. divisions,

$\therefore$,, ,, ,, 96°, ,, ,, $\frac{60}{360}\times 96$ or 16 min. divs.

[The remaining portion as before] Ans. $4\frac{4}{11}$ minutes past 4 and $39\frac{3}{11}$ minutes past 4 o'clock.

Ex. 25. If the radius (4000 miles) of the earth subtends an angle of 8·6″ at any point in the surface of the sun, what is the distance of the earth from the sun ? [C. U.]

Let S be a point on the surface of the sun and E be the centre of the earth. Then ES is the distance of the earth from the sun.

The angle ESP subtended by EP, the radius of the earth at the point S is 8·6″.

Now, suppose a circle to be drawn with centre S and radius SE. Then the arc cut off by the radius EP from this circle is roughly equal to EP, as $\angle$ESP is extremely small and ES is infinitely large.

Hence, $\because$ $\text{arc}=r\theta$, $\therefore$ $\frac{\text{arc}}{r}=\theta$,

$$\therefore \frac{\text{ER}}{\text{ES}}=\theta=\text{circular measure of } 8{\cdot}6''=\frac{8{\cdot}6\times\pi}{60\times 60\times 180} \text{ radian}$$

$$\therefore \frac{4000}{\text{ES}}=\frac{8{\cdot}6\times\pi}{60\times 60\times 180}=\frac{8{\cdot}6\times 22}{60\times 60\times 180\times 7}$$

$$\therefore \text{ ES or the reqd. distance}=\frac{4000\times 60\times 60\times 180\times 7}{8{\cdot}6\times 22}\text{ mi.}$$

$$=95898521 \text{ miles (App.)}$$

Ex. 26. Express in circular measure the angle between the two hands of a clock at 9-30 A. M. [C. U. '48]

At 9 o'clock the two hands are at right angles, *i. e.*, 15 min.-divisions apart, the hour-hand being at 12. At 9-30 the minute-hand having moved 30 min.-divisions is at the mark 6 and by this time the hour-hand has moved $\frac{30}{12}$ or $2\frac{1}{2}$ min.-divisions.

$\therefore$ at 9-30 A. M. the distance between the hands is $(15+2\frac{1}{2})$ or $17\frac{1}{2}$ min.-divisions

Now, if the hands are 15 min.-divisions apart, the included angle $=90^\circ$,

$\therefore$ If the hands are $17\frac{1}{2}$ min.-divisions apart,

the included angle $=\dfrac{90^\circ \times 17\frac{1}{2}}{15}=105^\circ=\dfrac{105\pi^c}{180}=\dfrac{7}{12}\pi^c$.

Exercise 1

1. Express in degrees, minutes and seconds :—

(i) $13214''$ (ii) $507{\cdot}5'$ (iii) $\cdot 24$ rt. angle.

2. Express the following in grades, minutes and seconds :

(i) $46235^{\backprime\backprime}$ (ii) $703{\cdot}3^{\backprime}$ (iii) $10{\cdot}7205^{g}$ (iv) $\cdot 03246$ rt. angle.

3. Express the following angles in terms of a right angle :—

(i) 30° (ii) $60^\circ 45'$ (iii) 135^{g} (iv) $360^\circ 20' 48''$ (v) $50^{g}35^{\backprime}24^{\backprime\backprime}$ (vi) $\dfrac{\pi^c}{3}$ (vii) $10\pi^c$.

4. Express the following angles in centesimal system :—

(i) $30^\circ 12' 36''$ (ii) $63^\circ 14' 51''$ (iii) $60^\circ 6' 45''$ (iv) $\dfrac{3\pi^c}{5}$ (v) $\dfrac{\pi^a}{12}$.

5. Reduce the following angles to sexagesimal measure :

(i) $40^{g}35^{\backprime}24^{\backprime\backprime}$ (ii) $60^{g}25^{\backprime\backprime}$ (iii) $40^{g}45^{\backprime}36^{\backprime\backprime}$ (iv) 2^c (v) $\frac{5}{9}\pi^c$ (vi) $\dfrac{2\pi^c}{3}$.

6. Express the following angles in radians :—

(i) 30° (ii) $75^\circ 30'$ (iii) $45^\circ 25' 36''$ (iv) 120^{g} (v) $70^{g} 40^{`}$ (vi) $325^{g} 25^{`} 72^{``}$.

7. The sum of two angles is 60 grades and their difference is 16° ; find the angles in degrees and grades.

8. The sum of two angles is 140° and their difference is 90^{g}, find the angles in degrees and radians.

9. Two angles are in the ratio of $5 : 3$ and their difference is (i) 40°, (ii) 100^{g}, (iii) $12\pi^{c}$, find the angles.

10. Two angles of a triangle are $48^\circ 47' 28''$ and $71^\circ 12' 32''$. Find the third angle in centesimal measure.

11. The sum of two angles is 152°. If the number of degrees in one is equal to the number of grades in the other, find the circular measure of the angles.

11. (*a*) The sum of three angles in G. P. is 171° and the number of grades in the greatest angle is to the number of circular units in the least as 450 is to π. Find the angles in grades.

12. Divide a right angle into two parts such that the number of degrees in the one and that of grades in the other may be in the ratio of $3 : 10$.

13. The angles of a triangle are in the ratio of $4 : 3 : 5$, find the circular measure of the least angle.

14. One angle of a triangle is $\frac{3\pi^{c}}{10}$ and another is 70^{g}. Express the third angle in degrees. [C. U.]

15. Divide $88^\circ 16'$ into two parts such that the number of Sexagesimal seconds in one part may be the same as the number of Centesimal seconds in the other.

16. The four angles of a quadrilateral are in A. P. ; the greatest angle is twice the least angle. Find the circular measure of the least angle. [B. U. 1897]

17. The angles of a quadrilateral are x°, 60°, 60^{g} and $\frac{5}{6}\pi^{c}$; find x. [B. U.]

18. Find the circular measure of an interior angle of a regular n-gon. [C. U.]

19. The difference of two angles is 1° and the sum of their circular measures is 1. Find the angles in degrees. [M. U.]

20. Show that the number of degrees in an angle of a regular dudecagon is equal to the number of grades in an angle of a regular octagon. [M. U.]

21. The angles of a pontagen are in A. P. and its greatest angle is thrice the least. Find the angles in degrees and radians. [B. U.]

22. The circumference of a circle is 176 yds.; find its radius. $(\pi = \frac{22}{7})$.

23. The radius of a circle is $7''$, find the angle subtended at the centre by an arc $11''$ in length.

24. Find the length of an arc which subtends $22\frac{1}{2}^\circ$ at the centre, the radius being $17{\cdot}6''$. $(\pi = \frac{22}{7})$.

25. Find the circumference of a coin whose diameter is one inch. $(\pi = \frac{22}{7})$.

26. An arc of a circle, measuring 2618 ft., subtends an angle of 60° at the centre. Find the radius of the circle. $(\pi = 3{\cdot}1416)$, [C. U.]

27. If an arc of 5 yds. 1 ft. 6 in. subtends an angle of $1{\cdot}8^c$ at the centre, find the radius of the circle in inches.

28. Assuming the distance of the sun from the earth to be 92,000,000 miles and the angle subtended by the diameter of the sun at the earth to be $32'$, find the diameter of the sun. $(\pi = \frac{22}{7})$.

29. An arc of a circle, whose radius is 48 ft., subtends an angle $66^\circ 15'$ at the centre. Find the length of the arc.

(Given circumference : diameter : : 333 : 106). [C. U.]

30. An arc of a circle, whose radius is 4000 miles, subtends at its centre an angle of $5''$. Find the length of the arc in miles. [C. U. 1877]

31. Assuming the radius of the earth to be 4000 miles, find the difference in latitude of two places, one of which is 100 miles north of the other. [$\pi = 3{\cdot}14159\cdots$]

32. Find, correct to a foot, the length of an arc of the Earth's equator, which subtends an angle of 1 minute at the centre of the Earth, supposing the radius of the Earth's equator to be 4000 miles and $\pi = 3{\cdot}14159$. [G. U. '50]

33. If the circumference of a circle be divided into five parts which are in A. P. and if the greatest part be six times the least, find in radians the angles subtended by the parts at the centre. [B. U.]

34. Calculate, correct to 3 places of decimals, the area of a circle of radius 6·28 ft. ($\pi = 3{\cdot}14159$). [E. B. S. B. '59]

35. A horse running along a circular path at the rate of 5 miles an hour traverses an arc which subtends 63° at the centre in 30 seconds. Find the diameter of the circle. ($\pi = \frac{22}{7}$).

36. The minute-hand of a clock is 2 ft. 6 in. long, how many inches does its extremity move in 30 minutes ? ($\pi = 3{\cdot}1416$)

37. Find the times between 5 and 6 o'clock when the angle between the two hands of a clock is 84°.

38. A regular polygon has three times as many sides as another and the number of degrees in each interior angle of the first is the same as the number of gardes in that of the second. Find the number of sides of each polygon.

39. The angles of a triangle are in A. P., and the greatest is double the least ; express the angles in radians. [C. U. '51]

40. Show that the number of degrees in an angle of a regular decagon is to the number of grades in an angle of a regular pentagon as 6 : 5. [C. U. '60]

41. Find the time between 1 P. M. and 2 P. M. when the angle between the hands of a clock is $186\frac{2}{3}$ grades. [C. U. '51]

42. A man $5\frac{1}{2}$ ft. high is seen from the distance of half-a-mile ; what is the angle that he subtends ? [W. B. S. F. '52]

43. Find the ratio of the radii of two circles at the centres of which two arcs of the same length subtend angles of 60° and 75°. [W. B. S. F. '53]

[**Hints.** $\because 180^\circ = \pi^c$, $\therefore 60^\circ = \frac{\pi^c}{3}$ and $75^\circ = \frac{5\pi^c}{12}$. If the radii of the two circles be r and R, then the arc of the first circle $= \frac{\pi}{3}.r$ and that of the second circle $= \frac{5\pi}{12}.R$.

Again, $\because$ the arcs are equal (hyp.),

$$\therefore \frac{\pi}{3}.r = \frac{5\pi}{12}R, \quad \therefore r : R = \frac{5\pi}{12} : \frac{\pi}{3} = 5 : 4.]$$

44. Express in circular measure an angle of a regular polygon of 10 sides. [U. U. '50]

Trigonometrical Ratios

12. **Definitions :** Let the $\angle$AOB be θ in magnitude. From any pt. P in OB, PM is drawn perpendicular to OA. Now with respect to the acute angle θ in the right-angled $\triangle$POM, PM is the perpendicular, OM the base and OP the hypotenuse. Again, with respect to the acute angle OPM, OM is the perpendicular, PM is the base and OP is the hypotenuse.

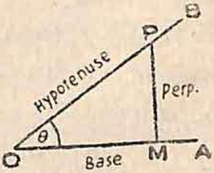


Fig. 5

Now, the trigonometrical ratios of the angle POM, *i.e.*, θ will be as follows :

$\frac{PM}{OP}\left(i.e., \frac{\text{perp.}}{\text{hyp.}}\right)$ is called the **Sine** of the angle θ.

$\frac{OM}{OP}\left(i.e., \frac{\text{base}}{\text{hyp.}}\right)$,, ,, **Cosine** ,, ,, ,,

$\frac{PM}{OM}\left(i.e., \frac{\text{perp.}}{\text{base}}\right)$,, ,, **Tangent** ,, ,, ,,

$\frac{OM}{PM}\left(i.e., \frac{\text{base}}{\text{perp.}}\right)$,, ,, **Cotangent** ,, ,,

$\frac{OP}{PM}\left(i.e., \frac{\text{hyp.}}{\text{perp.}}\right)$,, ,, **Cosecant** ,, ,,

$\frac{OP}{OM}\left(i.e., \frac{\text{hyp.}}{\text{base}}\right)$,, ,, **Secant** ,, ,, ,,

The ratios are briefly written as sin θ, cos θ, tan θ, cot θ, cosec θ, sec θ respectively.

Besides these, $1 - \cos\theta$ is called vers θ and $1 - \sin\theta$ is called co-vers θ.

13. The trigonometrical ratios are always the same for the same angle.

From any other point P_1 on the arm OB of $\angle$AOB draw P_1M_1 perpendicular to OA. It will be shown that the trigonometrical ratios of angle θ obtained from the $\triangle P_1OM_1$ will be the same as those obtained before from $\triangle$POM.

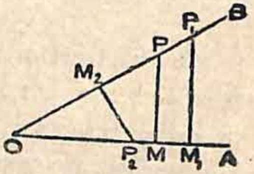


Fig. 6

In the $\triangle$s POM and P_1OM, $\angle$O is common, $\angle M = \angle M_1$ (being rt. $\angle$s), so the remaining angles P and P_1 are equal.

$\because$ the triangles are equiangular.

$$\therefore \quad \frac{PM}{OP} = \frac{P_1M_1}{OP_1}, \quad \frac{OM}{OP} = \frac{OM_1}{OP_1}, \quad \frac{PM}{OM} = \frac{P_1M_1}{OM_1}, \text{ etc.}$$

Thus it is proved that with respect to any position of P on OB, the trigonometrical ratios of $\angle$AOB are the same.

Similary, if P_2 be any point on OA and P_2M_2 be drawn perpendicular to OB, then the trigonometrical ratios of angle θ obtained from the $\triangle P_2OM_2$ will be the same as before, for the corresponding angles of the $\triangle POM$ and $\triangle P_2OM_2$ being equal their corresponding sides must be proportional.

14. Relation between the trigonometrical ratios.

It is evident from the definitions of the ratios that *sine* is the reciprocal of *cosecant*, *cosine* is the reciprocal of *secant* and *tangent* is the reciprocal of *cotangent*.

Thus, (i) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$.

$\therefore \quad \sin\theta \times \operatorname{cosec}\theta = 1.$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$, $\therefore \ \cos\theta \times \sec\theta = 1.$

(iii) $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$, $\therefore \ \tan\theta \times \cot\theta = 1.$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

[**N. B.** (i) The trigonometrical ratios are *all numbers* and hence the first four rules of arithmetic are applicable to them ; (ii) $\sin\ \theta$ signifies a number (it does not mean $\sin \times \theta$) ; (iii) the sum of $\sin\theta$ and $\cos\ \theta$ is $\sin\ \theta + \cos\ \theta$, just as the sum of the numbers a and b is $a+b$. Similary their difference is $\sin\ \theta - \cos\ \theta$, their product is $\sin\ \theta\ \cos\ \theta$ and their quotient is $\dfrac{\sin\theta}{\cos\theta}$. So $\sin\ \theta \times \sin\ \theta = (\sin\theta)^2$ written as $\sin^2\theta$, and $\sin\theta \times 2 = 2\sin\theta$. It is to be noticed that $(\sin\theta)^2$ is written as $\sin^2\theta$ but not as $\sin\theta^2$ which has a different meaning. Again $2\ \sin\ \theta$ and $\sin 2\theta$ are not the same, the one denotes twice the number $\sin\ \theta$ and the other denotes sine of twice the angle θ.]

Some Fundamental Relations

15. The following formulas are very important.

(i) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, (ii) $\sin^2+\cos^2\theta=1$,

(iii) $\sec^2\theta=1+\tan^2\theta$, (iv) $\operatorname{cosec}^2\theta=1+\cot^2\theta$.

(i) **Proof :**—[Draw the figure no. 5]

$\because$ $PM\perp OM$, $\therefore$ $\sin\theta=\dfrac{PM}{OP}$, $\cos\theta=\dfrac{OM}{OP}$ and $\tan\theta=\dfrac{PM}{OM}$.

Now, $\tan\theta=\dfrac{PM}{OM}=\dfrac{\frac{PM}{OP}}{\frac{OM}{OP}}$ (dividing the numerator and the denominator by OP)

$=\dfrac{\sin\theta}{\cos\theta}$.

(ii) Prove that $\sin^2\theta+\cos^2\theta=1$. [E. B. S. B. '30]

[In fig. no. 5] $\because$ $\angle M$ is a rt. angle, $\therefore$ $PM^2+OM^2=OP^2$;

$\therefore$ $\dfrac{PM^2}{OP^2}+\dfrac{OM^2}{OP^2}=\dfrac{OP^2}{OP^2}$ (dividing both sides by OP^2)

or, $\left(\dfrac{PM}{OP}\right)^2+\left(\dfrac{OM}{OP}\right)^2=1$; but $\sin\theta=\dfrac{PM}{OP}$ and $\cos\theta=\dfrac{OM}{OP}$,

$\therefore$ $\sin^2\theta+\cos^2\theta=1$.

(iii) Prove that $\sec^2\theta=1+\tan^2\theta$. [C.U. '49 ; E,B.S.B.'49]

[In fig. no. 5] $\because$ $\angle M$ is a right angle,

$\therefore$ $\sec\theta=\dfrac{OP}{OM}$, $\tan\theta=\dfrac{PM}{OM}$, and $OP^2=PM^2+OM^2$.

$\therefore$ $\dfrac{OP^2}{OM^2}=\dfrac{PM^2}{OM^2}+\dfrac{OM^2}{OM^2}$, or, $\left(\dfrac{OP}{OM}\right)^2=\left(\dfrac{PM}{OM}\right)^2+1$

$\therefore$ $\sec^2\theta=\tan^2\theta+1$.

(iv) In fig. no. 5, $\text{OP}^2=\text{PM}^2+\text{OM}^2$,

$$\text{and } \operatorname{cosec}\theta=\frac{\text{OP}}{\text{PM}},\ \cot\theta=\frac{\text{OM}}{\text{PM}}.$$

Now, $\because\ \text{OP}^2=\text{PM}^2+\text{OM}^2,\quad \therefore\ \frac{\text{OP}^2}{\text{PM}^2}=\frac{\text{PM}^2}{\text{PM}^2}+\frac{\text{OM}^2}{\text{PM}^2},$

or, $\left(\frac{\text{OP}}{\text{PM}}\right)^2=1+\left(\frac{\text{OM}}{\text{PM}}\right)^2,\quad \therefore\ \operatorname{cosec}^2\theta=1+\cot^2\theta.$

16. Some Identities

Ex. 1. Prove that $\sin^4\theta-\cos^4\theta+1=2\sin^2\theta$.

$$\begin{aligned}\sin^4\theta-\cos^4\theta+1&=(\sin^2\theta)^2-(\cos^2\theta)^2+1\\&=(\sin^2\theta+\cos^2\theta)(\sin^2\theta-\cos^2\theta)+1\end{aligned}$$

$=1\times(\sin^2-\theta\cos^2\theta)+\sin^2\theta+\cos^2\theta\quad[\because\ \sin^2\theta+\cos^2\theta=1]$

$=\sin^2\theta-\cos^2\theta+\sin^2\theta+\cos^2\theta=2\sin^2\theta.$

Ex. 2. Prove that $\cos \text{A}+\tan \text{A}\sin \text{A}=\sec \text{A}$. [C. U. '50]

$$\text{L. H. S.}=\cos \text{A}+\frac{\sin \text{A}}{\cos \text{A}}\times\sin \text{A}=\cos \text{A}+\frac{\sin^2 \text{A}}{\cos \text{A}}$$

$$=\frac{\cos^2 \text{A}+\sin^2 \text{A}}{\cos \text{A}}=\frac{1}{\cos \text{A}}=\sec \text{A}.$$

Ex. 3. Prove that $\frac{1}{\cot \text{A}+\tan \text{A}}=\sin \text{A}\cos \text{A}$. [C. U. '46]

$$\text{L. H. S.}=\frac{1}{\frac{\cos \text{A}}{\sin \text{A}}+\frac{\sin \text{A}}{\cos \text{A}}}=\frac{1}{\frac{\cos^2 \text{A}+\sin^2 \text{A}}{\sin \text{A}\cos \text{A}}}=\frac{1}{\frac{1}{\sin \text{A}\cos \text{A}}}$$

$$=\sin \text{A}\cos \text{A}.$$

Ex. 4. Prove that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta$.

$$\text{L. H. S.}=\sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}=\sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$=\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}=\frac{1-\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta-\tan\theta.$$

Ex. 5. Show that $\sec^2 A \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

[C. U. '46]

$$\text{L. H. S.} = (1+\tan^2 A)(1+\cot^2 A)$$
$$= 1 + \tan^2 A + \cot^2 A + \tan^2 A . \cot^2 A$$
$$= 1 + \tan^2 A + \cot^2 A + \tan^2 A \times \frac{1}{\tan^2 A} = 1 + \tan^2 A + \cot^2 A + 1$$
$$= \tan^2 A + \cot^2 A + 2.$$

Ex. 6. Show that $1 + \dfrac{\cot^2 A}{1+\operatorname{cosec} A} = \operatorname{cosec} A.$ [C. U. '47]

$$\text{L. H. S.} = 1 + \frac{\operatorname{cosec}^2 A - 1}{1+\operatorname{cosec} A} = 1 + \frac{(\operatorname{cosec} A + 1)(\operatorname{cosec} A - 1)}{\operatorname{cosec} A + 1}$$
$$= 1 + \operatorname{cosec} A - 1 = \operatorname{cosec} A.$$

Ex. 7. Show that $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta)(\tan\theta + \cot\theta) = 1.$ [C. U. '51]

$$\text{L. H. S.} = \left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$
$$= \frac{1-\cos^2\theta}{\cos\theta} \times \frac{1-\sin^2\theta}{\sin\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}$$
$$= \frac{\sin^2\theta}{\cos\theta} \times \frac{\cos^2\theta}{\sin\theta} \times \frac{1}{\sin\theta\cos\theta} = \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta} = 1.$$

Ex. 8. Prove that $\dfrac{1}{\sec\theta + \tan\theta} - \dfrac{1}{\cos\theta} = \dfrac{1}{\cos\theta} - \dfrac{1}{\sec\theta - \tan\theta}.$

[C. U. '44 ; G. U. '51]

$$\frac{1}{\sec\theta + \tan\theta} + \frac{1}{\sec\theta - \tan\theta} = \frac{\sec\theta - \tan\theta + \sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta}$$
$$= \frac{2\sec\theta}{1} = 2\sec\theta = \frac{2}{\cos\theta} = \frac{1}{\cos\theta} + \frac{1}{\cos\theta}.$$

Now by transposition we have

$$\frac{1}{\sec\theta + \tan\theta} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{\sec\theta - \tan\theta}.$$

Ex. 9. Prove that $\cos^2 A-\sin^2 A=\frac{1-\tan^2 A}{1+\tan^2 A}$. [G. U. '50]

L.H.S. $=\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$ $(\because \cos^2 A+\sin^2 A=1)$

$=\frac{\frac{\cos^2 A}{\cos^2 A}-\frac{\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A}+\frac{\sin^2 A}{\cos^2 A}}$ (dividing the numerator and the denominator by $\cos^2 A$)

$=\frac{1-\tan^2 A}{1+\tan^2 A}$.

Ex. 10. Show that $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\frac{\sin\theta+1}{\cos\theta}$.

L.H.S. $=\frac{(\tan\theta+\sec\theta)+(\tan^2\theta-\sec^2\theta)}{\tan\theta-\sec\theta+1}$

$[\because \sec^2\theta=1+\tan^2\theta, \therefore -1=\tan^2\theta-\sec^2\theta]$

$=\frac{(\tan\theta+\sec\theta)(1+\tan\theta-\sec\theta)}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$

$=\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}=\frac{\sin\theta+1}{\cos\theta}$.

11. If $\sin A+\sin^2 A=1$, prove that $\cos^2 A+\cos^4 A=1$. [E.B.S.B. '51]

Here, $\sin A+\sin^2 A=1$, $\therefore \sin A=1-\sin^2 A=\cos^2 A$.

$\therefore \sin^2 A=\cos^4 A$. Now, $\cos^2 A+\cos^4 A=\cos^2 A+\sin^2 A=1$.

Ex. 12. If $\cos A+\sin A=\sqrt{2}\cos A$, prove that

$\cos A-\sin A=\sqrt{2}\sin A$.

$\because \cos A+\sin A=\sqrt{2}\cos A\cdots(1)$,

$\therefore \cos^2 A+\sin^2 A+2\cos A\times\sin A=2\cos^2 A$ (squaring)

or, $\cos^2 A-\sin^2 A=2\cos A\sin A\cdots(2)$ [by transposition]

Now, from $(2)\div(1)$, $\cos A-\sin A=\frac{2\cos A\sin A}{\sqrt{2}\cos A}=\sqrt{2}\sin A$.

Ex. 13. Simplify $\cos^2 A \cos^2 B + \sin^2 A \sin^2 B + \cos^2 A \sin^2 B + \cos^2 B \sin^2 A$. [C. U. '43]

The given exp. $= (\cos^2 A \cos^2 B + \cos^2 A \sin^2 B) + (\sin^2 A \sin^2 B + \cos^2 B \sin^2 A)$

$$= \cos^2 A (\cos^2 B + \sin^2 B) + \sin^2 A (\sin^2 B + \cos^2 B)$$
$$= (\cos^2 B + \sin^2 B)(\cos^2 A + \sin^2 A) = 1 \times 1 = 1.$$

Ex. 14. Simplify $\dfrac{(\sin A + \cos A)(1 - \sin A \cos A)}{\sin^3 A + \cos^3 A}$

The given fraction

$$= \frac{(\sin A + \cos A)(\sin^2 A + \cos^2 A - \sin A \cos A)}{\sin^3 A + \cos^3 A}$$

$$= \frac{\sin^3 A + \cos^3 A}{\sin^3 A + \cos^3 A} = 1.$$

Ex. 15. Simplify $\cos^2 B \operatorname{cosec}^2 A - \sin^2 B \cot^2 A - \cos^2 B \cot^2 A + \sin^2 B \operatorname{cosec}^2 A$. [C. U. '45]

The given exp.

$$= \cos^2 B (\operatorname{cosec}^2 A - \cot^2 A) + \sin^2 B (\operatorname{cosec}^2 A - \cot^2 A)$$
$$= (\operatorname{cosec}^2 A - \cot^2 A)(\cos^2 B + \sin^2 B) = 1 \times 1 = 1.$$

Ex. 16. Simplify $\dfrac{\sec^4 A - 2 \sec^2 A \tan^2 A + \tan^4 A - 1}{\operatorname{cosec}^4 A - 2 \operatorname{cosec}^2 A \cot^2 A + \cot^4 A}$

[C. U. '41]

The fraction $= \dfrac{(\sec^2 A - \tan^2 A)^2 - 1}{(\operatorname{cosec}^2 A - \cot^2 A)^2} = \dfrac{(1)^2 - 1}{(1)^2} = \dfrac{1 - 1}{1} = 0.$

Ex. 17. Simplify $\dfrac{1}{1 + \sin^2 A} + \dfrac{1}{1 + \operatorname{cosec}^2 A}$.

The given exp.

$$= \frac{1}{1 + \sin^2 A} + \frac{1}{1 + \dfrac{1}{\sin^2 A}} = \frac{1}{1 + \sin^2 A} + \frac{1}{\dfrac{\sin^2 A + 1}{\sin^2 A}}$$

$$= \frac{1}{1 + \sin^2 A} + \frac{\sin^2 A}{1 + \sin^2 A} = \frac{1 + \sin^2 A}{1 + \sin^2 A} = 1.$$

Ex. 18. Simplify $\frac{\sin A-\sin B}{\cos A+\cos B}+\frac{\cos A-\cos B}{\sin A+\sin B}$. [C. U. '42]

$$\text{The given exp.}=\frac{\sin^2 A-\sin^2 B+\cos^2 A-\cos^2 B}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=\frac{(\sin^2 A+\cos^2 A)-(\sin^2 B+\cos^2 B)}{(\cos A+\cos B)(\sin A+\sin B)}$$

$$=\frac{1-1}{(\cos A+\cos B)(\sin A+\sin B)}=0.$$

Ex. 19. If $x\sin^3\alpha+y\cos^3\alpha=\sin\alpha\cos\alpha$, and $x\sin\alpha-y\cos\alpha=0$, then $x^2+y^2=1$. [C. U. '37 ; P. U. '43]

$\because\ x\sin\alpha-y\cos\alpha=0,\quad\therefore\ x\sin\alpha=y\cos\alpha$

$\therefore\ \frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=k$ (suppose) , $\therefore\ x=k\cos\alpha,\ y=k\sin\alpha$.

Now, $\because\ x\sin^3\alpha+y\cos^3\alpha=\sin\alpha\cos\alpha$

$\therefore\ k\cos\alpha\sin^3\alpha+k\sin\alpha\cos^3\alpha=\sin\alpha\cos\alpha,$

or, $k\sin\alpha\cos\alpha(\sin^2\alpha+\cos^2\alpha)=\sin\alpha\cos\alpha,$

or, $k\sin\alpha\cos\alpha=\sin\alpha\cos\alpha\ [\because\ \sin^2\alpha+\cos^2\alpha=1]$

$\therefore\ k=1,\quad\therefore\ x=\cos\alpha$ and $y=y=\sin\alpha$.

$\therefore\ x^2+y^2=\cos^2\alpha+\sin^2\alpha=1.$

Exercise 2

Prove the following :—

1. $\cos^4\theta-\sin^4\theta=2\cos^2\theta-1.$

2. $\frac{1}{\sin A\cos A}=\tan A\cot A.$ [E.B.S.B. '48]

3. $\cos^6 A+\sin^6 A+3\sin^2 A\cos^2 A=1.$

4. $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta.$ [C. U. '43]

5. $\tan^2 A\cot A+\cot^2 A\tan A=\sec A\operatorname{cosec} A.$

6. $(\sin\theta-\cos\theta)^2=1-2\sin\theta\cos\theta.$ [E.B.S.B. '50]

7. $\sec^4\theta-\sec^2\theta=\tan^4\theta+\tan^2\theta.$

8. $\sec^4 A-2\tan^2 A=1+\tan^4 A.$

9. $\operatorname{cosec}^4\theta-\cot^4\theta-1=2\cot^2\theta.$

10. $\sin^2\theta(1+\cot^2\theta)+\cos^2\theta(1+\tan^2\theta)=2.$ [C.U. '51]

11. $\sec^2 A\operatorname{cosec}^2 A=\tan^2 A+\cot^2 A+2.$ [C. U. '40]

12. $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}}=\cot\theta+\operatorname{cosec}\theta.$

13. $4\tan^2 A+3=3\sec^2 A+\tan^2 A.$

14. $1+\dfrac{\cos A}{\sec A}+\dfrac{\sin A}{\operatorname{cosec} A}=2.$

14. (*a*) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=2\cos^2\theta-1=1-2\sin^2\theta.$ [G. U. '52]

15. $\operatorname{cosec}^2 A\cos^2 B-\cot^2 A\sin^2 B+\operatorname{cosec}^2 A\sin^2 B$
$-\cot^2 A\cos^2 B=1.$

16. $\sqrt{\sec^2 A-1}=\sin A\sec A.$ **17.** $\dfrac{\tan\theta+\cot A}{\cot\theta+\tan A}=\cot A\tan\theta$

18. $\dfrac{\sec^4\theta-2\sec^2\theta\tan^2\theta+\tan^4}{\operatorname{cosec}^4\theta-2\operatorname{cosec}^2\theta\cot^2\theta+\cot^4\theta}=1.$ [C. U. '42]

19. $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A.$

20. $(\tan A+\sec A)^2=\dfrac{1+\sin A}{1-\sin A}.$ [C. U.]

21. $\dfrac{1}{\operatorname{cosec} A+\cot A}+\dfrac{1}{\operatorname{cosec} A-\cot A}=2\operatorname{cosec} A.$

22. $\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}-\dfrac{1}{\sin\theta}=\dfrac{1}{\sin\theta}-\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}.$

23. $(1+\tan^2 A)(1-\sin^2 A)=1.$ [C. U.]

24. $(1+\tan\theta-\sec\theta)(1+\cot\theta+\operatorname{cosec}\theta)=2.$

25. $\dfrac{1}{1+\cos^2 A}+\dfrac{1}{1+\sec^2 A}=1.$

26. $\dfrac{\cot A+1}{\cot A-1}+\dfrac{1+\tan A}{1-\tan A}.$ 27. $1+\dfrac{\tan^2\theta}{1+\sec\theta}=\sec\theta.$

28. $\dfrac{\cos A+\cos B}{\sin A+\sin B}=\dfrac{\sin A-\sin B}{\cos B-\cos A}.$

29. $\dfrac{\cos A}{1+\sin A}+\dfrac{1+\sin A}{\cos A}=2\sec A.$

30. $\dfrac{\cot\theta+\operatorname{cosec}\theta-1}{\cot\theta-\operatorname{cosec}\theta+1}=\dfrac{1+\cos\theta}{\sin\theta}.$

31. $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0.$

32. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A-1}+\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A+1}=2\sec^2 A.$ [W. B. S. F. '53]

Simplify :—

33. $\dfrac{\sin A}{\operatorname{cosec} A}+\dfrac{\cos A}{\sec A}+1.$ 34. $\dfrac{1-\tan\theta}{\cot\theta-1}-\dfrac{1+\tan\theta}{\cot\theta+1}.$

35. $\dfrac{\sec^4 A-2\sec^2 A\tan^2 A+\tan^4 A}{\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta}.$ [G. U. '49]

36. $\dfrac{1}{\operatorname{cosec} A-\cot A}+\dfrac{1}{\operatorname{cosec} A+\cot A}-\dfrac{2}{\sin A}.$

37. $\dfrac{\sin A+\sin B}{\cos A-\cos B}+\dfrac{\cos A+\cos B}{\sin A-\sin B}.$

38. $\dfrac{\sec A-\sec B}{\tan B+\tan A}+\dfrac{\tan B-\tan A}{\sec A+\sec B}.$

39. If $7\sin^2\theta+3\cos^2\theta=4$, show that $\tan\theta=\pm\dfrac{1}{\sqrt{3}}.$ [C. U. '38]

40. Prove that $\sin\theta(1+\tan\theta)+\cos\theta\,(1+\cot\theta)$ $=\sec\theta+\operatorname{cosec}\theta.$ [C. U. '35]

41. If $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$, then $\cos\theta+\sin\theta$ $=\sqrt{2}\cos\theta.$ [B. H. U. '46]

42. Show that the sum of $6\sin^4\theta-4\sin^6\theta$ and $6\cos^4\theta-4\cos^6\theta$ is the same for all values of θ.

17. To express the trigonometrical ratios in terms of any one of them.

(1) Express all the trigonometrical ratios is terms of the sine.

$$\because \ \sin^2\theta+\cos^2\theta=1, \ \therefore \ \cos^2\theta=1-\sin^2\theta,$$

$$\therefore \ \cos\theta=\sqrt{1-\sin^2\theta};$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}; \ \cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{\sqrt{1-\sin^2\theta}}{\sin\theta};$$

$$\sec\theta=\frac{1}{\cos\theta}=\frac{1}{\sqrt{1-\sin^2\theta}}; \ \operatorname{cosec}\theta=\frac{1}{\sin\theta}.$$

[**Otherwise**] In fig. no. 5, let $\angle AOP=\theta$, $OP=1$ (unit) and $PM=a$ units of length.

$$\therefore \ \sin\theta=\frac{PM}{OP}=\frac{a}{1}=a. \qquad \because \ OP^2=PM^2+OM^2,$$

$$\therefore \ OM^2=OP^2-PM^2=1-a^2, \ \therefore \ OM=\sqrt{1-a^2}.$$

$$\text{Now, } \cos\theta=\frac{OM}{OP}=\frac{\sqrt{1-a^2}}{1}=\sqrt{1-\sin^2\theta},$$

$$\tan\theta=\frac{PM}{OM}=\frac{a}{\sqrt{1-a^2}}=\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}, \text{ etc.}$$

[N. B. The sides of the triangle should be so taken that the value of the ratio in terms of which other ratios are to be expressed is a.]

(2) Express all trigonometrical ratios in terms of the **tangent.**

Draw fig. no. 5. Let $PM=a$ and $OM=1$; so $\tan\theta=\frac{PM}{OM}=a$.

$$\therefore \ OP^2=OM^2+PM^2, \ \therefore \ OP=\sqrt{OM^2+PM^2}=\sqrt{1+a^2}.$$

Now, $\sin\theta=\frac{\text{PM}}{\text{OP}}=\frac{a}{\sqrt{1+a^2}}=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$;

$\cos\theta=\frac{\text{OM}}{\text{OP}}=\frac{1}{\sqrt{1+a^2}}=\frac{1}{\sqrt{1+\tan^2\theta}}$; $\cot\theta=\frac{1}{\tan\theta}$;

$\sec\theta=\frac{1}{\cos\theta}=\sqrt{1+\tan^2\theta}$; $\text{cosec}\,\theta=\frac{1}{\sin\theta}=\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$.

18. Conversion of trigonometrical ratios.

Ex. 1. If $\sin \text{A}=\frac{1}{3}$, find cos A, tan A and sec A.

Here, $\sin\text{A}=\frac{\text{perp.}}{\text{hyp.}}=\frac{1}{3}$.

Draw fig no. 5. Let $\text{PM}=1$ unit of length and $\text{OP}=3$ units.

$\therefore\ \text{OM}^2=\text{OP}^2-\text{PM}^2=9-1=8,\quad\therefore\ \text{OM}=\sqrt{8}=2\sqrt{2}.$

Now, $\cos\text{A}=\frac{\text{OM}}{\text{OP}}=\frac{2\sqrt{2}}{3}$, $\tan\text{A}=\frac{\text{PM}}{\text{OM}}=\frac{1}{2\sqrt{2}}=\frac{1\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{4}$,

and $\sec\text{A}=\frac{1}{\cos\text{A}}=\frac{3}{2\sqrt{2}}=\frac{3\sqrt{2}}{4}$.

Ex. 2. If $\cos x=\frac{2}{3}$, find tan x, sec x and cosec x. [D.B. '48]

In the fig. no. 5, let $\text{OM}=2$ units and $\text{OP}=3$ units of length.

$\therefore\ \text{PM}^2=\text{OP}^2-\text{OM}^2=3^2-2^2=9-4=5,\quad\therefore\ \text{PM}=\sqrt{5}.$

Now, $\tan x=\frac{\text{PM}}{\text{OM}}=\frac{\sqrt{5}}{2}$; $\sec x=\frac{\text{OP}}{\text{OM}}=\frac{3}{2}$;

and $\text{cosec}\,x=\frac{\text{OP}}{\text{PM}}=\frac{3}{\sqrt{5}}$.

Ex. 3. The tangent of an angle is $\frac{3}{4}$; find the other trigonometrical ratios.

Here, $\tan\theta=\frac{\text{perp.}}{\text{base}}=\frac{3}{4}$. Draw fig. no. 5 and let $\text{PM}=3$ units and $\text{OM}=4$ units of length.

$\therefore\ \text{OP}^2=\text{PM}^2+\text{OM}^2=3^2+4^2=25,\quad\therefore\ \text{OP}=5.$

Now, $\sin\theta=\frac{\text{PM}}{\text{OP}}=\frac{3}{5}$, $\cos\theta=\frac{\text{OM}}{\text{OP}}=\frac{4}{5}$, $\cot\theta=\frac{\text{OM}}{\text{PM}}=\frac{4}{3}$,

$\sec\theta=\frac{\text{OP}}{\text{OM}}=\frac{5}{4}$ and $\operatorname{cosec}\theta=\frac{\text{OP}}{\text{PM}}=\frac{5}{3}$.

Ex. 4. If $\cos\theta=\frac{3}{5}$, where θ is a positive acute angle, find $\cot\theta$.

Here, $\cos\theta=\frac{\text{base}}{\text{hyp.}}=\frac{3}{5}$. Draw fig. no. 5, and let OM $=3$ units and OP $=5$ units of length. $\because$ $\text{PM}^2+\text{OM}^2=\text{OP}^2$,

$\therefore$ $\text{PM}^2=\text{OP}^2-\text{OM}^2=5^2-3^2=16$, $\therefore$ PM $=4$.

$\therefore$ $\cot\theta=\frac{\text{OM}}{\text{PM}}=\frac{3}{4}$.

Ex. 5. If cosine of an acute angle is $\sqrt{1-a^2}$, find its sine and show that its cotangent is $\frac{\sqrt{1-a^2}}{a}$.

Let the angle be θ, then $\cos\theta=\sqrt{1-a^2}$ (hypothesis).

$\because$ $\sin^2\theta+\cos^2\theta=1$, $\therefore$ $\sin^2\theta+(\sqrt{1-a^2})^2=1$,

or, $\sin^2\theta+1-a^2=1$, or, $\sin^2\theta=a^2$, $\therefore$ $\sin\theta=a$.

Again, $\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{\sqrt{1-a^2}}{a}$.

Ex. 6. If the tangent of an acute angle is c, prove that its sine $=\frac{c}{\sqrt{1+c^2}}$.

[C. U. '42]

Let the given angle be θ, then $\tan\theta=\frac{\text{perp.}}{\text{base}}=c$.

Draw fig. 5 and let PM $=c$ units and OM $=1$ unit of length.

$\therefore$ $\text{OP}^2=\text{PM}^2+\text{OM}^2=c^2+1$, $\therefore$ OP $=\sqrt{1+c^2}$.

Now, $\sin\theta=\frac{\text{PM}}{\text{OP}}=\frac{c}{\sqrt{1+c^2}}$.

[**Otherwise**] $\because\ \tan\theta = c, \quad \therefore\ \cot\theta = \frac{1}{c}.$

Now, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{1}{c^2} = \frac{c^2+1}{c^2},$

$\therefore\ \operatorname{cosec}\theta = \frac{\sqrt{1+c^2}}{c}, \quad \therefore\ \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{c}{\sqrt{1+c^2}}.$

Ex. 7. If $\tan\theta = \sqrt{3}$, show that $\sin\theta = \frac{\sqrt{3}}{2}.$

[See Ex. 3. Let $PM = \sqrt{3}$ units and $OM = 1$ unit.]

Exercise 3

1. Express all other trigonometrical ratios in terms of the cosine.

2. If $\sin A = \frac{3}{5}$, find tan A and sec A.

3. If $\cos A = \frac{12}{13}$, find cot A and cosec A.

4. If $\tan A = \frac{3}{4}$, find sin A and sec A.

5. If $\cot A = \sqrt{7}$, find the value of $\frac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}.$

6. If $\sec A = \frac{61}{11}$, find tan A and cosec A.

7. If $\tan\theta = \frac{1}{\sqrt{5}}$, prove that $\sin\theta = \frac{1}{\sqrt{6}}.$

8. If $\cos\theta = \frac{3}{5}$, show that $\tan\theta = \frac{4}{3}.$

9. If $\sec A = \frac{4}{3}$, show that $\cot A = \frac{3}{\sqrt{7}}.$

10. If $\cot A = \frac{x}{y}$, show that $\operatorname{cosec} A = \frac{\sqrt{x^2+y^2}}{y}.$

11. If $\operatorname{cosec} A = \frac{a}{b}$, show that $\tan A = \frac{b}{\sqrt{a^2-b^2}}.$

12. Sine of an acute angle is x. Prove that its cosine $= \sqrt{1-x^2}$ and its tangent $= \frac{x}{\sqrt{1-x^2}}.$ [C.U. '41 ; E.B.S.B. '51]

13. The tangent of an angle is $\frac{4}{3}$, find other direct trigonometrical ratios. [E. B. S. B. '50]

14. If $\cot A + \operatorname{cosec} A = 3$, find $\cos A$.

15. If $\tan A + \sec A = \dfrac{2}{\sqrt{3}}$, find $\sin A$.

16. If $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$, where A and B are positive acute angles, find the value of $\dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

[W. B. S. F. '58]

19. Trigonometrical ratios of certain angles.

(i) Trigonometrical ratios of 30°.

Let $\angle AOB = 30^\circ$ and $PM \perp OA$.

Then $\angle OPM = 60^\circ$. Produce PM to Q so that $MQ = PM$. Join QO.

In $\triangle POM$ and $\triangle QOM$, $PM = MQ$, OM is common to both and $\angle OMP = \angle OMQ$ (being rt. angles);

$\therefore$ the $\triangle$s are congruent.

$\therefore$ $\angle QOM = 30^\circ$, $\therefore$ $\angle POQ = 60^\circ$.

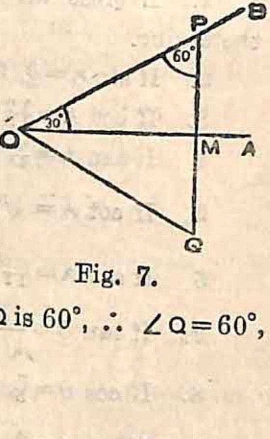


Fig. 7.

$\because$ in $\triangle OPQ$ each of the $\angle P$ and $\angle POQ$ is 60°, $\therefore$ $\angle Q = 60^\circ$,

$\therefore$ $\triangle POQ$ is equilateral.

$\therefore$ $PO = PQ = 2PM$, $\therefore$ $PM = \frac{1}{2}OP$.

Again, $OM^2 = OP^2 - PM^2 = OP^2 - (\frac{1}{2}OP)^2 = OP^2 - \frac{1}{4}OP^2 = \frac{3}{4}OP^2$,

$\therefore$ $OM = \dfrac{\sqrt{3}}{2}OP$. $\therefore$ $\sin 30^\circ = \dfrac{PM}{OP} = \dfrac{\frac{1}{2}OP}{OP} = \dfrac{1}{2}$;

$$\text{Hence, } \operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = \frac{1}{\frac{1}{2}} = 2\,;$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{\frac{\sqrt{3}}{2}OP}{OP} = \frac{\sqrt{3}}{2}\,;$$

Hence $\sec 30^\circ = \dfrac{1}{\cos 30^\circ} = \dfrac{1}{\frac{\sqrt{3}}{2}} = \dfrac{2}{\sqrt{3}}$;

$\tan 30^\circ = \dfrac{PM}{OM} = \dfrac{\frac{1}{2}OP}{\frac{\sqrt{3}}{2}OP} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$;

and $\cot 30^\circ = \dfrac{1}{\tan 30^\circ} = \dfrac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$.

(2) Ratios of 60°.

[In fig. 7] $\angle OPM = 60^\circ$; with respect to this angle, the perp. $= OM = \frac{\sqrt{3}}{2}OP$, base $= PM = \frac{1}{2}OP$ and hypotenuse $= OP$.

$\therefore \sin 60^\circ = \dfrac{OM}{OP} = \dfrac{\frac{\sqrt{3}}{2}OP}{OP} = \dfrac{\sqrt{3}}{2}$;

So, $\operatorname{cosec} 60^\circ = \dfrac{1}{\sin 60^\circ} = \dfrac{1}{\frac{\sqrt{3}}{2}} = \dfrac{2}{\sqrt{3}}$;

$\cos 60^\circ = \dfrac{PM}{OP} = \dfrac{\frac{1}{2}OP}{OP} = \dfrac{1}{2}$; hence, $\sec 60^\circ = \dfrac{1}{\cos 60^\circ} = \dfrac{1}{\frac{1}{2}} = 2$.

$\tan 60^\circ = \dfrac{OM}{PM} = \dfrac{\frac{\sqrt{3}}{2}OP}{\frac{1}{2}OP} = \sqrt{3}$; hence $\cot 60^\circ = \dfrac{1}{\sqrt{3}}$.

[N. B. $\sin 30^\circ = \cos 60^\circ$, $\cos 30^\circ = \sin 60^\circ$, $\tan 30^\circ = \cot 60^\circ$, $\cot 30^\circ = \tan 60^\circ$, $\operatorname{cosec} 30^\circ = \sec 60^\circ$, $\sec 30^\circ = \operatorname{cosec} 60^\circ$.]

(3) Ratios of 45°.

Draw fig. no. 5 and let $\angle AOB = 45^\circ$; from P in OB draw $PM \perp AO$. Then $\angle OMP = 90^\circ$, $\therefore$ the remaining $\angle OPM = 45^\circ = \angle AOB$. $\therefore$ $OM = PM$.

Again, $OP^2 = OM^2 + PM^2 = 2PM^2 = 2MO^2$.

$$\therefore \quad PM^2 = \tfrac{1}{2}OP^2, \quad \therefore \quad PM = \frac{1}{\sqrt{2}}OP \text{ and } OM = \frac{1}{\sqrt{2}}OP.$$

$$\therefore \quad \sin 45^\circ = \frac{PM}{OP} = \frac{\frac{1}{\sqrt{2}}OP}{OP} = \frac{1}{\sqrt{2}}\text{; hence, } \operatorname{cosec} 45^\circ = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{\frac{1}{\sqrt{2}}OP}{OP} = \frac{1}{\sqrt{2}}\text{; hence, } \sec 45^\circ = \sqrt{2}.$$

$$\tan 45^\circ = \frac{PM}{OM} = \frac{OM}{OM} = 1\text{; hence } \cot 45^\circ = 1.$$

(4) Ratios of 0°.

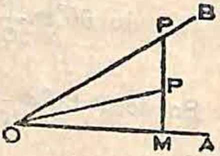


Fig. 8

Let $\angle AOB$ be a very small positive angle. Draw $PM \perp OA$ from any pt. P in OB. As $\angle AOB$, i.e., $\angle POM$ becomes less and less, PM and OP also become less and less. Thus when $\angle POM$ becomes smaller than any quantity we can assign, *i.e.*, when it becomes zero, OP coincides with OM and then PM vanishes (*i.e.*, $PM = 0$) and $OP = OM$.

$$\therefore \quad \sin 0^\circ = \frac{PM}{OP} = \frac{0}{OP} = 0\text{; } \cos 0^\circ = \frac{OM}{OP} = \frac{OM}{OM} = 1\text{;}$$

$\tan 0^\circ = \frac{PM}{OM} = \frac{0}{OM} = 0$; $\cot 0^\circ =$ the value of $\frac{OM}{PM}$ where OM is a finite quantity and PM is infinitely small (*i.e.*, 0) $= \infty$ (an infinitely large quantity). $\therefore$ $\cos 0^\circ = \infty$ (infinity).

Similarly $\operatorname{cosec} 0^\circ = \frac{OP}{PM} = \infty$; $\sec 0^\circ = \frac{OP}{OM} = \frac{OM}{OM} = 1$

(5) Ratios of 90°.

Let $\angle AOB$ be an acute angle and let PM be drawn perpendicular to OA from any point P on OB. As this angle gradually increases, OM gradually decreases and OP comes nearer to OX. Thus when OB is perpendicular to OA, *i. e.*, when OP actually describes a right angle, then M coincides with O and OP and PM are equal.

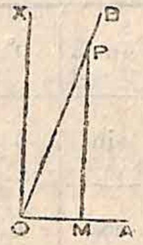


Fig. 9

$$\therefore \quad OM = 0.$$

$$\text{Now, } \sin 90^\circ = \frac{PM}{OP} = \frac{OP}{OP} = 1\ ;\ \cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0.$$

$$\tan 90^\circ = \frac{PM}{OM} = \frac{\text{a finite quantity}}{\text{an infinitely small quantity}} = \infty.$$

$$\cot 90^\circ = \frac{OM}{PM} = \frac{0}{PM} = 0\ ;\ \sec 90^\circ = \frac{OP}{OM} = \infty,$$

$$\text{and } \operatorname{cosec} 90^\circ = \frac{OP}{PM} = \frac{PM}{PM} = 1.$$

[**N. B.** In case of tan 90°, OM approaches 0 and in the limit $OM = 0$. $\therefore \tan 90^\circ = \frac{\text{a finite quantity}}{\text{zero}} = \infty$.

Strictly speaking no *finite quantity* can be divided by zero. Hence the values of cot 0°, tan 90°, sec 90°, cosec 0° cannot be determined. So they are undefined.

Now, if originally the $\angle AOB$ be obtuse and if it approaches 90°, then OM is negative and it will be zero in the limit. PM is positive. $\therefore \tan 90^\circ = -\infty$. Hence we may write $\tan 90^\circ = \pm\infty$. This applies to sec 90°, cot 0°, cosec 0° also.]

20. The following table should be committed to memory :—

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

[**N. B.** If only sine and cosine of an angle are known, the other ratios can easily be reproduced. It may be noted that if only sine and cosine of the angles 0°, 30° and 45° be committed to memory, then the whole table may be reproduced. There is an artifice for remembering the sines. Write the numbers 0, 1, 2, 3, 4 and divide each by 4 and then take the square roots of the quotients.

These square roots, *i. e.*, $\sqrt{\frac{0}{4}}$, $\sqrt{\frac{1}{4}}$, $\sqrt{\frac{2}{4}}$, $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{4}{4}}$ are respectively the sines of angles 0°, 30°, 45°, 60° and 90°.]

Trigonometrical ratios of Complementary angles.

In fig. 5, $\angle P$ is the complement of angle θ.

We have $\sin \angle P = \frac{OM}{OP} = \cos \theta$,

$$\therefore \quad \sin(90^\circ - \theta) = \cos\theta, \ \cos(90^\circ - \theta) = \sin\theta,$$
$$\tan(90^\circ - \theta) = \cot\theta, \ \cot(90^\circ - \theta) = \tan\theta,$$
$$\sec(90^\circ - \theta) = \operatorname{cosec}\theta, \ \operatorname{cosec}(90^\circ - \theta) = \sec\theta.$$

Some examples worked out

Ex. 1. Verify that $\sin^2 30^\circ + \sin^2 45^\circ + \cos^2 60^\circ = 1$.

$$\text{L. H. S.} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4} = 1.$$

Ex. 2. Show that $\cos 60^\circ = 1 - 2\sin^2 30^\circ$. [E.B.S.B. '49]

$\text{L.H.S.} = \cos\ 60^\circ = \tfrac{1}{2}$;

Again, $1 - 2\sin^2 30^\circ = 1 - 2 \times (\tfrac{1}{2})^2 = 1 - 2 \times \tfrac{1}{4} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$.

$\therefore\ \cos 60^\circ = 1 - 2\sin^2 30^\circ$.

Ex. 3. Prove that $\dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ} = \sqrt{3}$. [C. U. '40]

$$\text{L.H.S.} = \frac{2 \times \dfrac{1}{\sqrt{3}}}{1 - \left(\dfrac{1}{\sqrt{3}}\right)^2} = \frac{\dfrac{2}{\sqrt{3}}}{1 - \dfrac{1}{3}} = \frac{\dfrac{2}{\sqrt{3}}}{\dfrac{2}{3}} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}.$$

Ex. 4. Find the numerical value of $\tan^2 30^\circ + 2\sin 60^\circ + \tan 45^\circ - \tan 60^\circ + \cos^2 30^\circ$. [G. U. '48]

$$\text{The given exp.} = \left(\frac{1}{\sqrt{3}}\right)^2 + 2 \times \frac{\sqrt{3}}{2} + 1 - \sqrt{3} + \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \tfrac{1}{3} + \sqrt{3} + 1 - \sqrt{3} + \tfrac{3}{4} = \tfrac{1}{3} + 1 + \tfrac{3}{4} = \tfrac{25}{12} = 2\tfrac{1}{12}.$$

Ex. 5. Find the numerical value of $\sin^3 30^\circ + 4\cot^2 45^\circ - \sec^2 60^\circ$. [C. U. '50]

The given exp. $= (\tfrac{1}{2})^3 + 4.\ (1)^2 - (2)^2 = \tfrac{1}{8} + 4 - 4 = \tfrac{1}{8}$.

Ex. 6. Find the numerical value of

$\tan^2 45^\circ.\ \sin 60^\circ.\ \tan 30^\circ.\ \tan^2 60^\circ$. [C. U. '49]

$$\text{The given exp.} = (1)^2.\left(\frac{\sqrt{3}}{2}\right).\ \frac{1}{\sqrt{3}}.\ (\sqrt{3})^2 = 1 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} \times 3 = 1\tfrac{1}{2}.$$

Ex. 7. Find the simplest value of

$3\tan^2 45^\circ - \sin^2 60^\circ - \tfrac{1}{2}\cot^2 30^\circ + \tfrac{1}{8}\sec^2 45^\circ$. [C. U. '51]

$$\text{The given exp.} = 3.(1)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 - \tfrac{1}{2}.(\sqrt{3})^2 + \tfrac{1}{8}(\sqrt{2})^2$$
$$= 3 - \tfrac{3}{4} - \tfrac{3}{2} + \tfrac{1}{4} = 1.$$

Ex. 8. If $\tan\theta=\frac{3}{4}$, find the other trigonometrical ratios.

Let $\angle A$ of the $\triangle AOB$ be a right angle and $\angle O=\theta$.

$\because$ $\tan\theta=\frac{3}{4}$, $\therefore$ if prep. $AB=3$ units of length, then the base $AO=4$ units of length.

$\therefore$ $OB^2=3^2+4^2=25$, $\therefore$ $OB=\sqrt{25}=5$.

Now, $\sin\theta=\frac{AB}{OB}=\frac{3}{5}$, $\cos\theta=\frac{OA}{OB}=\frac{4}{5}$, $\cot\theta=\frac{1}{\frac{3}{4}}=\frac{4}{3}$,

$\operatorname{cosec}\theta=\frac{1}{\frac{3}{5}}=\frac{5}{3}$ and $\sec\theta=\frac{1}{\frac{4}{5}}=\frac{5}{4}$.

Ex. 9. Prove that $\sqrt{\frac{1+\cos 30^\circ}{1-\cos 30^\circ}}=\sec 60^\circ+\tan 60^\circ$.

[C. U. '42]

$$\text{L.H.S.}=\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}}=\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}=\sqrt{\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})}}$$

$$=\sqrt{\frac{(2+\sqrt{3})^2}{1}}=2+\sqrt{3}.$$

Again, R. H. S. $=2+\sqrt{3}$. $\therefore$ both the sides are equal.

Ex. 10. Show that $\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\tan\frac{\pi}{4}=\sin\frac{\pi}{6}\cos\frac{\pi}{3}+\sin\frac{\pi}{2}$.

Here the angles are in circular measure.

$\because$ π radians $=180^\circ$,

$\therefore$ L. H. S. $=\sin 60^\circ\cos 30^\circ+\sin 30^\circ\tan 45^\circ$

$$=\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\frac{1}{2}\times 1=\frac{3}{4}+\frac{1}{2}=\frac{5}{4}=1\tfrac{1}{4},$$

and R.H.S. $=\sin 30^\circ\cos 60^\circ+\sin 90^\circ=\frac{1}{2}\times\frac{1}{2}+1=\frac{1}{4}+1=1\frac{1}{4}$.

$\therefore$ both the sides are equal.

Ex. 11. Show that $\frac{1+2\sin 60^\circ\cos 60^\circ}{\sin 60^\circ+\cos 60^\circ}+\frac{1-2\sin 60^\circ\cos 60^\circ}{\sin 60^\circ-\cos 60^\circ}$
$=2\cos 30^\circ$.

[C. U. '41]

$$\text{L.H.S.}=\frac{1+2.\frac{\sqrt{3}}{2}.\frac{1}{2}}{\frac{\sqrt{3}}{2}+\frac{1}{2}}+\frac{1-2.\frac{\sqrt{3}}{2}.\frac{1}{2}}{\frac{\sqrt{3}}{2}-\frac{1}{2}}=\frac{1+\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2}}+\frac{1-\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}-1}{2}}$$

$$=\frac{\frac{2+\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2}}+\frac{\frac{2-\sqrt{3}}{2}}{\frac{\sqrt{3}-1}{2}}=\frac{2+\sqrt{3}}{\sqrt{3}+1}+\frac{2-\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{(2+\sqrt{3})(\sqrt{3}-1)+(2-\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{2\sqrt{3}+3-2-\sqrt{3}+2\sqrt{3}-3+2-\sqrt{3}}{3-1}=\frac{2\sqrt{3}}{2}=\sqrt{3}.$$

R. H. S. $=2\cos 30^\circ=2\times\frac{\sqrt{3}}{2}=\sqrt{3}$. $\therefore$ both sides are equal.

Ex. 12. Simplify :—

$$\frac{2\tan^2 30^\circ}{1-\tan^2 30^\circ}+(\sec^2 45^\circ-\cot^2 45^\circ)-(\sin^2 30^\circ+\sin^2 60^\circ).\ \text{[G.U. '50]}$$

$$\text{The given exp.}=\frac{2\left(\frac{1}{\sqrt{3}}\right)^2}{1-\left(\frac{1}{\sqrt{3}}\right)^2}+\{(\sqrt{2})^2-(1)^2\}-\left\{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}$$

$$=\frac{\frac{2}{3}}{1-\frac{1}{3}}+\{2-1\}-\{\tfrac{1}{4}+\tfrac{3}{4}\}=\frac{\frac{2}{3}}{\frac{2}{3}}+1-1=\frac{2\times3}{3\times2}=1.$$

Exercise 4

1. Prove that :—

(1) $\cos 60^\circ=\frac{1}{2}$ and $\tan 30^\circ=\frac{1}{\sqrt{3}}$ [C. U. '41]

(2) $\tan 60^\circ=\sqrt{3}$ and $\operatorname{cosec} 30^\circ=2$ [C. U. '43, '45]

(3) $\cot 30^\circ=\sqrt{3}$ and $\sin 90^\circ=1$ [C. U. '44]

(4) $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ (5) $\sin 45^\circ=\frac{1}{\sqrt{2}}$ [G. U. '51]

2. Find the value of $\cos 60^\circ$, $\sec 30^\circ$ and $\cot 60^\circ$. [C.U. '47]

3. Find the value of $4 \sin^2 45^\circ + \tan^2 60^\circ + \operatorname{cosec}^2 30^\circ$. [C. U. '40]

4. Find the numerical value of $\cot^2 \frac{\pi}{4} + \sin^3 \frac{\pi}{6} - \cos^3 \frac{\pi}{3}$.

5. Find the simplest value of $4 \sin^2 30^\circ + 2 \cos^2 45^\circ - 3 \cos^2 60^\circ$.

6. Simplify $\dfrac{\sec^2 45^\circ - \cot^2 45^\circ}{\sin^2 60^\circ + \cos^2 30^\circ} + \dfrac{\sin^2 90^\circ - \cos^2 60^\circ}{\frac{1}{4} \tan^2 30^\circ}$.

7. Prove that $\tan 30^\circ . \sin 60^\circ + \tan 60^\circ . \cos 30^\circ = 2$.

7. (*a*) Show that $\tan^2 \frac{\pi}{4} . \sin \frac{\pi}{3} . \tan \frac{\pi}{6} . \tan^2 \frac{\pi}{3} = 1\frac{1}{2}$.

8. Prove that $\sin . \frac{\pi}{3} . \tan \frac{\pi}{6} + \sin \frac{\pi}{2} . \cos \frac{\pi}{3} = 2 \sin^2 \frac{\pi}{4}$.

9. Show that $\sqrt{\dfrac{1 + \sin 30^\circ}{1 - \sin 30^\circ}} = \sec 60^\circ \sin 60^\circ$.

10. Prove that

$$\frac{1 + 2 \sin 60^\circ \cos 60^\circ}{\sin 60^\circ + \cos 60^\circ} + \frac{1 - 2 \sin 60^\circ \cos 60^\circ}{\sin 60^\circ - \cos 60^\circ} = 2 \cos 30^\circ.$$

[C. U. '41]

11. Find the value of $\cot^2 30^\circ - 2 \cos^2 60^\circ - \frac{3}{4} \sec^2 45^\circ - 4 \sin^2 30^\circ$. [W.B.S.F. '52]

12. Find the value of $\dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} + \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$. [W.B S.F. '52]

13. Find the value of $\dfrac{1 - \sin^2 30^\circ}{1 + \sin^2 45^\circ} \times \dfrac{\cos^2 60^\circ + \cos^2 30^\circ}{\operatorname{cosec}^2 90^\circ - \cot^2 90^\circ} \div (\sin 60^\circ \tan 30^\circ)$. [G. U. '54]

21. Elimination :—

Ex. 1. Eliminate A from the equations $x = a \sin A$ and $y = b \cos A$.

$\because x = a \sin A$, $\therefore \frac{x}{a} = \sin A$; and $\because y = b \cos A$, $\therefore \frac{y}{b} = \cos A$.

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 A + \cos^2 A = 1$. $\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ex. 2. Eliminate B from $x=a\sin B$, $y=\dfrac{a}{\sec B}$.

$$\because\quad x=a\sin B,\quad \therefore\quad \frac{x}{a}=\sin B.$$

Again, $\because\ y=\dfrac{a}{\sec B}$,

$$\therefore\ \frac{y}{a}=\frac{1}{\sec B}=\cos B.$$

Now, $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}=\sin^2 B+\cos^2 B=1$, $\therefore\ x^2+y^2=a^2$.

Ex. 3. If $x=a\sec\theta$, $y=b\tan\theta$, prove that $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

[G. U. '51]

$$\because\quad x=a\sec\theta,\quad \therefore\ \frac{x}{a}=\sec\theta,\ \therefore\quad \frac{x^2}{a^2}=\sec^2\theta\ ;$$

$$\because\quad y=b\tan\theta,\quad \therefore\ \frac{y}{b}=\tan\theta,\ \therefore\quad \frac{y^2}{b^2}=\tan^2\theta,$$

Now, $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\sec^2\theta-\tan^2\theta=1+\tan^2\theta-\tan^2\theta=1$.

$$\therefore\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Ex. 4. If $\sin\theta+\cos\theta=a$ and $\tan\theta+\cot\theta=b$, show that $b(a^2-1)=2$.

$\tan\theta+\cot\theta=b$, or, $\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=b$, or, $\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=b$,

$$\text{or,}\quad \frac{1}{\sin\theta\cos\theta}=b,\quad \therefore\ \sin\theta\cos\theta=\frac{1}{b}\cdots(1).$$

Again, $\because\ \sin\theta+\cos\theta=a$, $\therefore\ \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=a^2$,

or, $1+2\sin\theta\cos\theta=a^2$, or, $2\sin\theta\cos\theta=a^2-1\cdot\cdot\ (2)$

$\therefore$ From (1) and (2) we have

$$2\times\frac{1}{b}=a^2-1,\quad \text{or,}\quad \frac{2}{b}=a^2-1,\ \therefore\ b(a^2-1)=2.$$

Ex. 5. If $\sin A+\cos A=m$, and $\sec A+\operatorname{cosec} A=n$, prove that $n(m^2-1)=2m$. [B. U.]

$$n(m^2-1)=(\sec A+\operatorname{cosec} A)\{(\sin A+\cos A)^2-1\}$$

$$=\left(\frac{1}{\cos A}+\frac{1}{\sin A}\right)(\sin^2 A+\cos^2 A+2\sin A\cos A-1)$$

$$=\left(\frac{\sin A+\cos A}{\cos A\sin A}\right)(1+2\sin A\cos A-1)$$

$$=\frac{m}{\sin A\cos A}\times 2\sin A\cos A=2m.$$

Ex. 6. If $\tan A+\sin A=m$, and $\tan A-\sin A=n$, prove that $m^2-n^2=4\sqrt{mn}$. [H. S. 64 (Compl.) ; C. U. '14]

Here $m+n=\tan A+\sin A+\tan A-\sin A=2\tan A$,

and $m-n=2\sin A$.

Again, $mn=(\tan A+\sin A)(\tan A-\sin A)=\tan^2 A-\sin^2 A$

$$=\frac{\sin^2 A}{\cos^2 A}-\sin^2 A=\sin^2 A\left(\frac{1}{\cos^2 A}-1\right)=\sin^2 A\left(\frac{1-\cos^2 A}{\cos^2 A}\right)$$

$$=\sin^2 A\times\frac{\sin^2 A}{\cos^2 A}=\sin^2 A\tan^2 A.\quad\therefore\ \sqrt{mn}=\sin A\tan A.$$

Now, $m^2-n^2=(m+n)(m-n)$

$$=2\tan A.2\sin A=4\tan A\sin A=4\sqrt{mn}.$$

Ex. 7. If $\sin A=m$ and $\tan A=n$, prove that $\frac{1}{m^2}-\frac{1}{n^2}=1$.

Here $\frac{1}{\sin A}=\frac{1}{m}$ and $\frac{1}{\tan A}=\frac{1}{n}$,

$$\therefore\ \frac{1}{\sin^2 A}=\frac{1}{m^2}\text{ and }\frac{1}{\tan^2 A}=\frac{1}{n^2}.$$

$$\therefore\ \frac{1}{\sin^2 A}-\frac{1}{\tan^2 A}=\frac{1}{m^2}-\frac{1}{n^2},\text{ or, }\operatorname{cosec}^2 A-\cot^2 A=\frac{1}{m^2}-\frac{1}{n^2},$$

$$\text{or, }1+\cot^2 A-\cot^2 A=\frac{1}{m^2}-\frac{1}{n^2},\quad\therefore\ \frac{1}{m^2}-\frac{1}{n^2}=1.$$

Exercise 5

Eliminate θ from the following equations :—

1. $x = a \sin\theta$ and $y = b\cos\theta$.

2. $x = b\cos\theta$, $y = b\sin\theta$.

3. $x = a(\sec\theta + \tan\theta)$, $y = a(\sec\theta - \tan\theta)$.

4. $a = b(\operatorname{cosec}\theta + \cot\theta)$, $c = b(\operatorname{cosec}\theta - \cot\theta)$.

5. $x = a\tan\theta$, $y = b\cot\theta$.

6. If $\sin A = \dfrac{n}{\sqrt{m^2+n^2}}$, show that $n\cos A = m\sin A$.

7. If $\cos\theta = p$ and $\cot\theta = q$, show that $\dfrac{1}{p^2} - \dfrac{1}{q^2} = 1$.

8. If $\cos\theta = \dfrac{p^2-q^2}{p^2+q^2}$, show that $\operatorname{cosec}\theta + \cot\theta = \dfrac{p}{q}$.

[**Hints** :—Putting the value of $\cos\theta$ in $\sin\theta = \sqrt{1-\cos^2\theta}$ we get $\sin\theta = \dfrac{2pq}{p^2+q^2}$. Then $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta} \cdots$]

9. If $\sin\theta - \cos\theta = 1$, show that $\sin\theta + \cos\theta = \pm 1$.

10. If $\sin\theta = \dfrac{2mn}{m^2+n^2}$, show that $\sec\theta + \tan\theta = \dfrac{m+n}{m-n}$.

22. Solution of equations.

Ex. 1. Solve $2\cos\theta = \sec\theta$.

$2\cos\theta = \sec\theta$, or, $2\cos\theta = \dfrac{1}{\cos\theta}$, or, $2\cos^2\theta = 1$,

or, $\cos^2\theta = \frac{1}{2}$, or, $\cos\theta = \dfrac{1}{\sqrt{2}}$; but $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$,

$\therefore\ \cos\theta = \cos 45^\circ$, $\therefore\ \theta = 45^\circ$.

[**N. B. Magnitudes of trigonometrical ratios** : In the relation $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta$ and $\cos^2\theta$ are square numbers and hence neither $\sin^2\theta$ nor $\cos^2\theta$ can be negative. Now, since the sum of $\sin^2\theta$ and $\cos^2\theta$ is 1, neither $\sin^2\theta$ nor

$\cos^2\theta$ can be greater than 1, as in that case one of them must be less than 1 (*i. e.*, negative, which is impossible). Thus whatever may be the value of θ, $\sin\theta$ and $\cos\theta$ can never be numerically greater than 1 (*i.e.*, $\sin\theta$ and $\cos\theta$ can not have values like 2, -2, $+3{\cdot}5$). Thus $\sin\theta$ and $\cos\theta$ must lie between $+1$ and -1. Consequently $\sec\theta$ and $\operatorname{cosec}\theta$ which are reciprocals of $\cos\theta$ and $\sin\theta$ respectively can never be numerically less than 1. Tan θ and $\cot\theta$ can have any value greater or less than 1 according to the value of θ.]

Ex. 2. If $2\sin A = 2 - \cos A$, find $\sin A$.

$2\sin A = 2 - \cos A$,

or, $2\sin A - 2 = -\cos A$, or, $(2\sin A - 2)^2 = (-\cos A)^2$,

or, $4\sin^2 A - 8\sin A + 4 = \cos^2 A = 1 - \sin^2 A$,

or, $5\sin^2 A - 8\sin A + 3 = 0$,

or, $5\sin^2 A - 5\sin A - 3\sin A + 3 = 0$,

or, $(\sin A - 1)(5\sin A - 3) = 0$, $\therefore$ $\sin A = 1$, or, $\frac{3}{5}$.

Ex. 3. If $2\sin^2\theta = 3\cos\theta$; find θ, if it be a positive acute angle. [C. U. '50]

$2\sin^2\theta = 3\cos\theta$, or, $2(1-\cos^2\theta) = 3\cos\theta$,

or, $2\cos^2\theta + 3\cos\theta - 2 = 0$,

or, $2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$,

or, $(2\cos\theta - 1)(\cos\theta + 2) = 0$, $\therefore$ $\cos\theta = \frac{1}{2}$ or -2.

$\because$ $\cos\theta$ cann ot be numerically less than -1, $\therefore$ $\cos\theta \neq -2$.

$\therefore$ $\cos\theta = \frac{1}{2} = \cos 60^\circ$. $\therefore$ $\theta = 60^\circ$.

Ex. 4. Solve $\sin\theta + \operatorname{cosec}\theta = \frac{7}{2\sqrt{3}}$, if θ be a positive acute angle.

$\sin\theta + \operatorname{cosec}\theta = \frac{7}{2\sqrt{3}}$, or, $\sin\theta + \frac{1}{\sin\theta} = \frac{7}{2\sqrt{3}}$,

or, $\sin^2\theta + 1 = \frac{7\sin\theta}{2\sqrt{3}}$ [multiplying by $\sin\theta$]

or, $2\sqrt{3}\sin^2\theta + 2\sqrt{3} = 7\sin\theta$, or, $2\sqrt{3}\sin^2\theta - 7\sin\theta + 2\sqrt{3} = 0,$

or, $2\sqrt{3}\sin^2\theta - 4\sin\theta - 3\sin\theta + 2\sqrt{3} = 0,$

or, $2\sin\theta(\sqrt{3}\sin\theta - 2) - \sqrt{3}(\sqrt{3}\sin\theta - 2) = 0,$

or, $(2\sin\theta - \sqrt{3})(\sqrt{3}\sin\theta - 2) = 0,$

$\therefore$ $\sin\theta = \frac{\sqrt{3}}{2}$ or $\frac{2}{\sqrt{3}}$. $\because$ $\sin\theta$ cannot exceed $+1$,

$\therefore$ $\sin\theta \neq \frac{2}{\sqrt{3}}$, $\therefore$ here $\sin\theta = \frac{\sqrt{3}}{2} = \sin 60^\circ$, $\therefore$ $\theta = 60^\circ$.

Ex. 5. Solve $\cos 12A = \sin 6A$.

$\because$ The sine of an angle = the cosine of its complement,

$\therefore$ We have $\cos 12A = \sin 6A = \cos(90^\circ - 6A)$

$\therefore$ $12A = 90^\circ - 6A$, or, $18A = 90^\circ$, $\therefore$ $A = 5^\circ$.

Ex. 6. If $r\cos\theta = 2\sqrt{3}$ and $r\sin\theta = 2$, where θ is an acute angle find r and θ. [C. U. '45 ; G. U. '49]

$\because$ $r\cos\theta = 2\sqrt{3}\cdots(1)$ and $r\sin\theta = 2\cdots(2)$,

$\therefore$ From $(1)\div(2)$ we have $\frac{r\cos\theta}{r\sin\theta} = \frac{2\sqrt{3}}{2}$ or, $\frac{\cos\theta}{\sin\theta} = \sqrt{3}$,

or, $\cot\theta = \sqrt{3} = \cot 30^\circ$. $\therefore$ $\theta = 30^\circ$.

Now from (1) we have $r\cos 30^\circ = 2\sqrt{3}$, or, $r.\frac{\sqrt{3}}{2} = 2\sqrt{3}$.

$\therefore$ $r = \frac{2\sqrt{3}\times 2}{\sqrt{3}} = 4$. Hence, $r = 4$ and $\theta = 30^\circ$.

[If in the sum $r\cos\theta = 2$, $r\sin\theta = 2\sqrt{3}$ as in G. U. '52, then, $r = 4$, $\theta = 60^\circ$.]

Ex. 7. If $\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \tan 60^\circ$, find the value of x. [C. U. '49]

From the given equation we have

$$(1)^2 - (\tfrac{1}{2})^2 = x.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\sqrt{3}, \quad \text{or,} \quad 1 - \tfrac{1}{4} = x.\frac{\sqrt{3}}{2},$$

or, $\frac{3}{4} = x.\frac{\sqrt{3}}{2}$, $\therefore$ $x = \frac{3}{4}\times\frac{2}{\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$.

Ex. 8. Solve $\dfrac{\sin A+\cos A}{\sin A-\cos A}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

By componendo & dividendo we have

$$\frac{2\sin A}{2\cos A}=\frac{2\sqrt{3}}{2},\quad \text{or,}\quad \frac{\sin A}{\cos A}=\sqrt{3},$$

or, $\tan A=\sqrt{3}=\tan 60^\circ$. $\therefore\ A=60^\circ$.

Ex. 9. If $1+\sin^2 A=3\sin A\cos A$, find $\tan A$ and deduce that one value of $\sin A=\dfrac{1}{\sqrt{2}}$. [C. U. '50]

$1+\sin^2 A=3\sin A\cos A$,

or. $\cos^2 A+\sin^2 A+\sin^2 A=3\sin A\cos A$,

or, $\cos^2 A-3\sin A\cos A+2\sin^2 A=0$,

or, $\cos^2 A-2\sin A\cos A-\sin A\cos A+2\sin^2 A=0$,

or, $(\cos A-2\sin A)(\cos A-\sin A)=0$,

$\therefore$ either, $\cos A-2\sin A=0\cdots(1)$ or, $\cos A-\sin A=0\cdots(2)$.

From (1) we have $\cos A=2\sin A$,

or, $\dfrac{1}{2}=\dfrac{\sin A}{\cos A}$ (dividing by $2\cos A$), $\therefore\ \tan A=\frac{1}{2}$.

From (2) we have $\cos A=\sin A$, or, $1=\dfrac{\sin A}{\cos A}=\tan A$.

$$\therefore\ \tan A=1,\ \text{or,}\ \tfrac{1}{2}.$$

Again, $*\because\ \cos A=\sin A$, $\therefore\ A=45^\circ$ [$\because$ sine and cosine of 45° are equal], $\therefore\ \sin A=\sin 45^\circ=\dfrac{1}{\sqrt{2}}$.

[***Otherwise** : $\sin A=\cos A=\sin(90^\circ-A)$.

$\therefore\ A=90^\circ-A$, or, $2A=90^\circ$, $\therefore\ A=45^\circ$;

$\therefore\ \sin A=\sin 45^\circ=\dfrac{1}{\sqrt{2}}$.]

Exercise 6

Solve the following :—

1. $3 \tan \theta = \cot \theta$.
2. $4 \cos A = \sec A$.
3. $\tan A + \cot A = \frac{4}{\sqrt{3}}$.
4. $\sin A + \cos A = 1$.
5. $\sec \theta + \cos \theta = \frac{5}{2}$.
6. $2 \sin \theta = \operatorname{cosec} \theta$.
7. $\sin \theta + \operatorname{cosec} \theta = \frac{3}{\sqrt{2}}$.
8. $\cot 5\theta = \tan 10\theta$.
9. $\sin 10\theta = \cos 8\theta$.

10. If $\tan A + \cot A - 2 = 0$, find $\sin A$.

11. If $x \tan \theta = \sqrt{3}$ and $x \cot \theta = 3\sqrt{3}$, find x and θ, θ being an acute angle.

12. If $p \sin \theta = \sqrt{3}$ and $p \cos \theta = 1$, where θ is an acute angle, find p and θ.

13. If $x \sin 30^\circ \tan 60^\circ \cos 30^\circ = \sin^2 60^\circ - \sec^2 45^\circ$, find the value of x.

14. Given $\cot^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{3} = x \cot \frac{\pi}{3} \sin \frac{\pi}{4} \cos \frac{\pi}{4}$, find x.

15. Is there any value of θ which can make $\operatorname{cosec} \theta = \frac{x^2 - y^2}{x^2 + y^2}$, x and y being two real and unequal numbers ?

16. Given that $2(\cos^2\theta - \sin^2\theta) = 1$ where θ is a positive acute angle, prove that $\cot \theta = \sqrt{3}$. [C. U. '51]

17. Solve $\frac{\sec \theta + \tan \theta}{\sec \theta - \tan \theta} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$.

18. Solve $2 \sin^2\theta + 3 \cos \theta = 3$. [Pat. U. '50]

19. Solve $\sec A + \tan A = \sqrt{3}$.

Practical applications of Trigonometry
(Height and Distance)

23. One of the applications of Trigonometry is to ascertain the heights and distances of distant (or inaccessible) objects by observing the angles that are subtended by those objects at the eye of the observer. The instruments Theodolite and Sextant are used in such practical works.

24. Angle of elevation and Angle of depression

A straight line parallel to the surface of the earth is called a **horizontal** line and a straight line perpendicuiar to it is called a **vertical** line.

If OX be a horizontal line drawn through O, the eye of the observer, and A be a point above OX, then the angle which the st. line joining A and O makes with OX (*i.e.*, ∠AOX) is called the **angle of elevation** of A as seen from O.

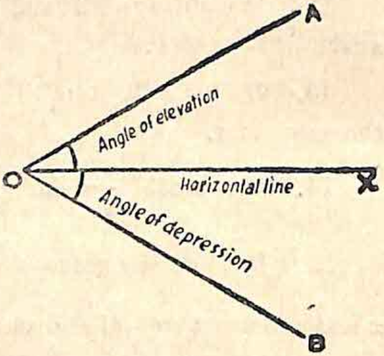


Fig. 9

If the point be at B below the horizontal line OX, then ∠XOB is called the **angle of depression** of B as seen from O.

Ex. 1. From a point O, at a horizontal distance of 100 yards from the bottom of the cliff PM, the angle of elevation of the top of the cliff is observed to be 30°. The height of the cliff is to be determined.

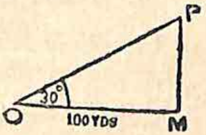


Fig. 10

Here PM is the height,
$OM = 100$ yds. and $\angle POM = 30°$.

$\therefore$ In the right-angled $\triangle$POM,

$$\frac{PM}{OM} = \tan \angle POM = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\therefore \frac{PM}{100} = \frac{1}{\sqrt{3}}, \therefore PM = \frac{100}{\sqrt{3}} \text{ yds.} = \frac{100 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \text{ yds.} = \frac{100\sqrt{3}}{3} \text{ yds.}$$
$$= 57{\cdot}7 \text{ yds. (App.)}.$$

Ex. 2. The angle of elevation of the top of a chimney at a distance of 200 ft. is 60°. Find the height of the chimney. [C. U. 1927]

Let PM be the height of the vertical chimney and let the angle of elevation of P from the point O at a horizontal distance of 200 ft. from the chimney be 60° (*i.e.*, the $\angle POM = 60^\circ$).

Now, $\frac{PM}{OM} = \tan 60^\circ$, *or*, $\frac{PM}{200} = \sqrt{3}$,

$\therefore$ $PM = 200\sqrt{3}$ ft. $= 346{\cdot}4$ ft.

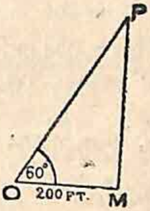


Fig. **11**

Ex. 3. A tree on the bank of a river is 50 ft. high and the angle of elevation of the top of the tree from a point on the opposite bank is observed to be 30°. Find the breadth of the river.

[Draw the fig. no. 11] Let PM be a tree, 50 ft. high, on the bank of a river and let the angle of elevation of P, observed from the pt. O on the other bank just opposite to the tree, be 30°. Then $\angle POM = 30^\circ$.

It is required to find OM, the breadth of the river.

Here, $\frac{PM}{OM} = \tan 30^\circ = \frac{1}{\sqrt{3}}$, or, $\frac{50}{OM} = \frac{1}{\sqrt{3}}$, $\therefore$ $OM = 50\sqrt{3}$ ft.

Ex. 4. Find the length of the shadow of a pillar 60 ft. high on the horizontal plane through its foot, when the angle of elevation of the sun is 60°.

[Draw the fig. no. 12] Let PM be a pillar 60 ft. high and the sun be at S. Let OM be the shadow of PM, the rays of the sun coming along SPO. The angle of elevation of the sun is 60° observed from the pt. O.

Now, $\frac{PM}{OM} = \tan 60°$, or, $\frac{60}{OM} = \sqrt{3}$, or, $OM\sqrt{3} = 60$,

$$\therefore \quad OM = \frac{60}{\sqrt{3}} = 20\sqrt{3}.$$

∴ The length of the shadow $= 20\sqrt{3}$ ft. $= 34{\cdot}64$ ft.

Ex. 5. Find the angle of elevation of the sun when the shadow of a pole 9 ft. high is $3\sqrt{3}$ ft. long. [C. U. '46]

Let PM be the pole 9 ft. high and S be the position of the sun. Let the rays of the sun coming along the line SPO cast the shadow OM of the pole PM.

Here $OM = 3\sqrt{3}$ ft. and $\angle POM$ is the angle of elevation of the sun.

Now, $\tan POM = \frac{PM}{OM} = \frac{9}{3\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$;

but $\tan 60° = \sqrt{3}$, $\quad \therefore \quad \angle POM = 60°$.

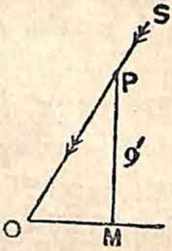


Fig. 12

Ex. 6. A vertical chimney subtends angles of 30° and 60° at points A and B which lie in a horizontal line through the base of the chimney. Find its height if AB is 100 ft. [C. U. '40]

Let PM be the height of the chimney, $\angle PAM = 30°$, and $\angle PBM = 60°$ and $AB = 100$ ft.

Now, $\frac{AM}{PM} = \cot 30° = \sqrt{3} \cdots (1)$

and $\frac{BM}{PM} = \cot 60° = \frac{1}{\sqrt{3}} \cdots (2)$

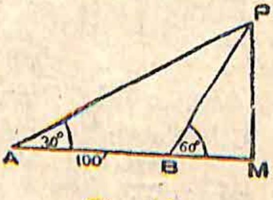


Fig. 13

∴ From (1) − (2) we have

$$\frac{AM - BM}{PM} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}},$$

or, $\frac{100}{PM} = \frac{2}{\sqrt{3}}$, or, $2PM = 100\sqrt{3}$, $\quad \therefore \; PM = 50\sqrt{3}$ ft.

∴ The height of the chimney $= 50\sqrt{3}$ ft. $= 86{\cdot}6$ ft.

Ex. 7. The shadow of a chimney is found to diminish 50 ft. in length when the angles of elevation of the sun changes from 30° to 60°. Find the height of the chimney. [C.U. '43]

[Draw the fig. 13] Let AM be the shadow of the chimney PM and let $\angle$PAM, the angle of elevation of the sun from A, be 30°. Again, let the $\angle$PBM, the angle of elevation of the sun from B, be 60°, when BM is the shadow of PM. BM is less than AM by 50 ft., *i.e.*, AB = 50 ft.

To find the height of the chimnny.

Now, $\frac{AM}{PM} = \cot 30° = \sqrt{3}$, and $\frac{BM}{PM} = \cot 60° = \frac{1}{\sqrt{3}}$.

$\therefore$ (subtracting) $\frac{AM - BM}{PM} = \sqrt{3} - \frac{1}{\sqrt{3}}$,

or, $\frac{50}{PM} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$, or, $2PM = 50\sqrt{3}$, $\therefore$ $PM = 25\sqrt{3}$.

$\therefore$ the required height of the chimney $= 25\sqrt{3}$ feet.

Ex. 8. The angular elevations of the top of a tower from two points in the same horizontal line with its foot are observed to be θ and ϕ respectively. Find the distance between the two points of observation, if the height of the tower is h. [C. U. '42]

The two points of observation A and B may be (i) on the same side of the tower, or (ii) on the opposite sides of it.

Let h be the height of the tower and the angles of elevation of P from the points A and B in both cases be θ and ϕ respectively. To find the distance AB.

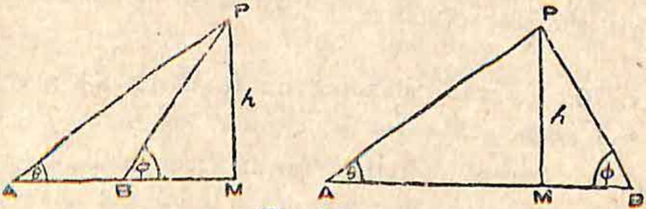


Fig. 14

In both the figures $\frac{AM}{PM} = \cot\theta \cdots (1)$ and $\frac{BM}{PM} = \cot\phi \cdots\cdots (2)$

In the first case, subtracting (2) from (1) we have $\frac{AM - BM}{PM}$ $= \cot\theta - \cot\phi$, or, $\frac{AB}{h} = \cot\theta - \cot\phi$, $\therefore$ $AB = h(\cot\theta - \cot\phi)$.

In the second case, adding (1) and (2) we have $\frac{AM+BM}{PM}$ $=\cot\theta+\cot\phi$, or, $\frac{AB}{h}=\cot\theta+\cot\phi$, $\therefore$ $AB=h(\cot\theta+\cot\phi)$.

Ex. 9. A post stands on the top of a pillar 40 ft. high. The elevations of the tops of the pillar and the post are respectively 30° and 45° to an observer standing on the horizontal line from the foot of the pillar. Find the length of the post and the distance of the observer from the pillar.

Suppose the post PQ stands on the pillar PM, 40 ft. high. Let PAM and QAM, the angles of elevation of P and Q from A, be 30° and 45° respectively. To find PQ and AM.

Now, $\frac{QM}{AM}=\tan 45^\circ=1$, $\therefore$ $QM=AM$.

Again, $\frac{PM}{AM}=\tan 30^\circ=\frac{1}{\sqrt{3}}$, or, $\frac{40}{AM}=\frac{1}{\sqrt{3}}$,

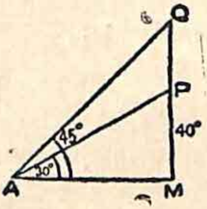


Fig. 15

$\therefore$ $AM=40\sqrt{3}$, $\therefore$ $PQ=QM-PM=AM-PM=40\sqrt{3}-40$
$=40(\sqrt{3}-1)=40(1{\cdot}732-1)=29{\cdot}28$ ft.

$\therefore$ the required distance $=40\sqrt{3}$ ft. $=62{\cdot}28$ ft. and the length of the post $=29{\cdot}28$ ft.

Ex. 10. A man, standing on the bank of a river, observes that the angle subtended by a tower, just on the opposite bank is 60°, but when he retires 60 ft. from the bank he finds the angle to be 30°; find the height of the tower and the breadth of the river.

[Draw the fig. 13] In the figure let PM be the height of the tower and BM the breadth of the river.

Then $\angle PBM=60^\circ$. The man retires 60 ft. behind from B to A and observes that $\angle PAM$ is 30°.

Now, $\frac{AM}{PM} = \cot 30^\circ = \sqrt{3}$, and $\frac{BM}{PM} = \cot 60^\circ = \frac{1}{\sqrt{3}}$

$\therefore$ (Subtracting) $\frac{AM - BM}{PM} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$, or, $\frac{60}{PM} = \frac{2}{\sqrt{3}}$,

$\therefore$ $PM = 30\sqrt{3}$, $\therefore$ the height of the tower $= 30\sqrt{3}$ ft.

$= 30 \times 1.732$ ft. $= 51.96$ ft.

Again, $\frac{BM}{PM} = \frac{1}{\sqrt{3}}$, $\therefore$ $BM = \frac{1}{\sqrt{3}} \times PM = \frac{1}{\sqrt{3}} \times 30\sqrt{3} = 30$.

$\therefore$ the breadth of the river $= 30$ ft.

Ex. 11. From an aeroplane vertically above a straight road the angles of depression of two consecutive mile-stones on the road are observed to be 45° and 30°. Find the height of the aeroplane above the road. [C. U. '**41** ; G. U. '49]

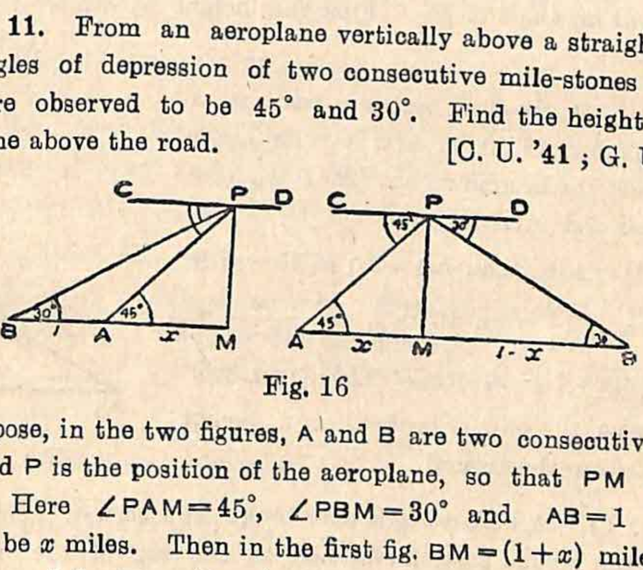


Fig. 16

Suppose, in the two figures, A and B are two consecutive mile posts and P is the position of the aeroplane, so that PM is its height. Here $\angle PAM = 45^\circ$, $\angle PBM = 30^\circ$ and $AB = 1$ mile. Let AM be x miles. Then in the first fig. $BM = (1 + x)$ miles and in the second fig. $BM = (1 - x)$ mile.

Now, $\frac{PM}{AM} = \tan 45^\circ = \mathbf{1}$, or, $\frac{PM}{x} = 1$, $\therefore$ $PM = x$.

In the first figure, $\frac{PM}{BM} = \tan 30^\circ = \frac{1}{\sqrt{3}}$, or, $\frac{x}{1+x} = \frac{1}{\sqrt{3}}$,

or, $x\sqrt{3} = 1 + x$, or, $x\sqrt{3} - x = 1$, or, $x(\sqrt{3} - 1) = 1$,

$\therefore$ $x = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{3 - 1} = \frac{\sqrt{3} + 1}{2} = 1.37$ (App.).

In the second figure, $\frac{\text{PM}}{\text{BM}} = \tan 30^\circ = \frac{1}{\sqrt{3}}$, or, $\frac{x}{1-x} = \frac{1}{\sqrt{3}}$,

or, $x\sqrt{3} = 1 - x$, or, $x\sqrt{3} + x = 1$, or, $x(\sqrt{3}+1) = 1$,

$\therefore \quad x = \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2} = \cdot 37$ (App.).

$\therefore$ The height of the aeroplane = 1·37 miles, or, ·37 mile.

Ex. 12. A vertical post 15 ft. high is broken at a certain height, and its upper part, not completely separated, meets the ground at an angle of 30°. Find the height at which the post is broken. [G. U. '51]

Let PM be the vertical post at M which is broken at Q and let its upper part meet the ground at A. Then we have PQ = AQ and ∠QAM = 30°.

Let QM = x ft., then AQ = PQ = $(15 - x)$ ft.

Now, $\frac{\text{QM}}{\text{AQ}} = \sin 30^\circ = \frac{1}{2}$, or, $\frac{x}{15-x} = \frac{1}{2}$,

or, $2x = 15 - x$, or, $3x = 15$, $\therefore x = 5$.

Hence, the post is broken at a height of 5 ft. from the ground.

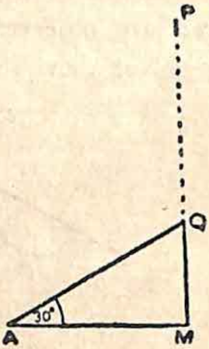


Fig. 17

Ex. 13. A long upright pole broke at a certain height, one end of the broken portion was tied to the top of the unbroken part, the other end lying on the ground at a distance of 20 ft. from the pole. The angle of inclination of this transverse portion was measured and found to be 30°. How high was the pole originally ? [E. B. S. B. '49]

[Draw the fig. of Ex. 12.] Let the pole PM standing on the ground at M be broken at Q and let its upper end P meet the ground at A. Then, by the problem, AM = 20 ft., ∠QAM = 30°, and PM = AQ + QM.

Now, $\frac{\text{QM}}{\text{AM}} = \tan 30^\circ = \frac{1}{\sqrt{3}}$, or, $\frac{\text{QM}}{20} = \frac{1}{\sqrt{3}}$, or, $\text{QM} = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}$

Again, $\frac{\text{AM}}{\text{AQ}} = \cos\ 30^\circ = \frac{\sqrt{3}}{2}$,

or, $\frac{20}{\text{AQ}} = \frac{\sqrt{3}}{2}$, or, $\text{AQ}\sqrt{3} = 40$, or, $\text{AQ} = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}$,

∴ The required height of the pole

$$= \text{AQ} + \text{QM} = \left(\frac{40\sqrt{3}}{3} + \frac{20\sqrt{3}}{3}\right)\text{ft.} = \frac{60\sqrt{3}}{3}\text{ ft.}$$

$$= 20\sqrt{3}\text{ ft.} = 20 \times 1{\cdot}732\text{ ft.} = 34{\cdot}64\text{ ft.}$$

Ex. 14. From the top of a cliff, 150 ft. high, the angles of depression of the top and bottom of a pillar are found to be 30° and 60° respectively ; find the height of the pillar.

Let the cliff PM and the pillar AB be respectively 150 ft. and x feet high. Draw PC ∥ MA. Then we have $\angle\text{CPB} = 30^\circ$ and $\angle\text{CPA} = 60^\circ$. Suppose AB produced meets PC at C, so that $\text{BC} = \text{AC} - \text{AB} = \text{PM} - \text{AB} = 150 - x$.

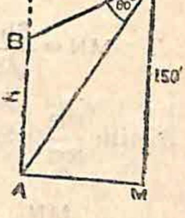


Fig. 18

Now, $\angle\text{PAM} = \angle\text{APC} = 60^\circ$,

$$\therefore\ \frac{\text{AM}}{\text{PM}} = \cot\ 60^\circ = \frac{1}{\sqrt{3}},$$

$$\therefore\ \text{AM} = \frac{\text{PM}}{\sqrt{3}} = \frac{150}{\sqrt{3}} = \frac{150\sqrt{3}}{3} = 50\sqrt{3}.$$

Again, $\frac{\text{BC}}{\text{AM}} = \frac{\text{BC}}{\text{PC}} = \tan\ 30^\circ = \frac{1}{\sqrt{3}}$,

or, $\frac{150 - x}{50\sqrt{3}} = \frac{1}{\sqrt{3}}$ [∵ $\text{AM} = 50\sqrt{3}$], or, $150 - x = 50$,

∴ $x = 100$. ∴ the height of the pillar = 100 ft.

Ex. 15. From the top and bottom of a cliff the angles of depression and elevation of the top of a pillar 60 ft. high are observed to be 30° and 60° respectively. Find the height of the cliff.

In the figure let h and 60 ft. be respeciively the heights of the cliff PM and the pillar QN.

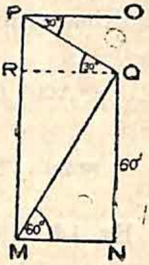


Fig. 19

Let the angle of depression of the pt. Q from the pt. P be 30° and the angle of elevation of Q from M be 60°.

Draw the horizontal line PO and $QR \perp PM$. Now, $\angle OPQ = 30^\circ$ and $\angle QMN = 60^\circ$. $\because$ OP, QR and NM are perpendiculars on the same straight line PM, $\therefore$ they are parallel. Then $\angle PQR =$ alt. $\angle OPQ = 30^\circ$, and $\angle RQM =$ alt. $\angle QMN = 60^\circ$. Now, $RM = QN = 60'$.

$$\frac{QN}{MN} = \tan 60' = \sqrt{3}, \text{ or, } \frac{60}{MN} = \sqrt{3},$$

$$\therefore \quad MN = \frac{60}{\sqrt{3}} = \frac{60\sqrt{3}}{3} = 20\sqrt{3}.$$

$$\text{Again, } \frac{PR}{RQ} = \tan 30^\circ = \frac{1}{\sqrt{3}}, \text{ or, } \frac{PR}{MN} = \frac{1}{\sqrt{3}} \ [\because RQ = MN],$$

$$\therefore \quad PR = \frac{MN}{\sqrt{3}} = \frac{20\sqrt{3}}{\sqrt{3}} = 20. \therefore PM = PR + RM = 20' + 60' = 80'.$$

$\therefore$ The height of the cliff = 80 ft.

Ex. 16. A man finds that at a point due south of a tower the angle of elevation of its top is 60°, he then walks 200 ft. due east on a horizontal plane and finds that the angle of elevation is 30° ; find the height of the tower.

In the diagram, N is a point due south of the tower PM. The angle of elevation of the top of the tower from N is 60°. From N the man walks 200 ft. to R due east of N and finds the angle of elevation of P to be 30°.

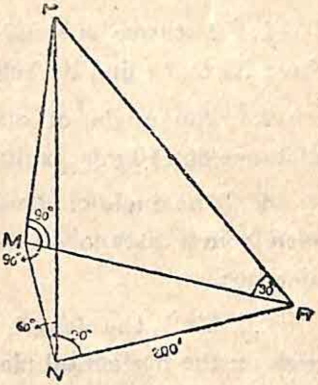


Fig. 20

So here $\angle MNP = 60^\circ$, $\angle MRP = 30^\circ$, $\angle MNR = 90^\circ$, $\angle PMR = 90^\circ$, $\angle PMN = 90^\circ$. Now, from $\triangle PMN$,

$$\frac{PM}{MN} = \tan 60^\circ = \sqrt{3}, \quad \therefore \; MN = \frac{PM}{\sqrt{3}}.$$

Again, $\dfrac{PM}{MR} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$,

$\therefore \; MR = PM\sqrt{3}$.

Since, $\angle MNR$ is a right angle, $\therefore \; MR^2 = MN^2 + NR^2$,

$$\text{or, } (PM\sqrt{3})^2 = \left(\frac{PM}{\sqrt{3}}\right)^2 + (200)^2, \text{ or, } 3PM^2 = \frac{PM^2}{3} + 200^2,$$

$$\text{or, } 8PM^2 = 3 \times 200^2, \text{ or, } PM^2 = \frac{3 \times 200^2}{8}.$$

$$\therefore \; PM = \frac{\sqrt{3} \times 200}{2\sqrt{2}} = \frac{100\sqrt{3}}{\sqrt{2}} = \frac{100\sqrt{6}}{2} = 50\sqrt{6}$$
$$= 50 \times 2{\cdot}44949 \cdots = 122{\cdot}47 \cdots$$

$\therefore$ The height of the tower $= 122{\cdot}47$ ft. (App.).

Ex. 17. From the top of a tree 20 ft. high the angles of elevation and depression of the top and base of a temple situated on the same horizontal plane are observed to be 45° and 30° respectively. Find the height of the temple. [U. U. '51]

[**Hints :** (Vide fig. 19). Suppose QN is a tree 20 ft. high, PM a temple and from Q, the angle of elevation of P (*i.e.*, $\angle PQR$) is 45° and the angle of depression of M (*i. e.*, $\angle RQM$) is 30°.

Then $\dfrac{QN}{MN} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$, or, $\dfrac{20}{MN} = \dfrac{1}{\sqrt{3}}$,

$\therefore \; MN = 20\sqrt{3}$. $\therefore \; RQ = MN = 20\sqrt{3}$,

$\therefore \; PR = MN = 20\sqrt{3}$ ($\because \; \angle PQR = 45^\circ = \angle RPQ$).

$\therefore \; PM = MR + PR = (20 + 20\sqrt{3})$ ft. $= (20 + 34{\cdot}64)$ ft.
$= 54{\cdot}64$ ft. (App.).

Exercise 7

1. A tower subtends an angle of 30° at a distance of 300 ft. from its foot ; find its height.

2. The angle of elevation of the top of a monument at a distance of 150 yds. is 60°. Find the height of the monument.

3. The angle of elevation of the top of a chimney 60 ft. high seen from a distance in the same horizontal plane is 30°, find the distance.

4. Find the length of the shadow of a vertical stick 6 ft. high on the horizontal plane through its foot, when the sun is at an altitude of 30°. [G. U. '50]

5. The shadow of a pole is $3\sqrt{3}$ ft. long when the angle of elevation of the sun is 60°. Find the height of the pole.

6. A ladder 30 ft. long reaches the top of a wall and makes an angle of 45° with the ground ; find the height of the wall.

7. What is the angle of elevation of the sun when the shadow of a post 24 ft. high is $24\sqrt{3}$ ft. long ?

8. Find the height of a chimney when it is found that on walking towards it 50 ft. in a horizontal line through its base the angular elevation changes from 30° to 45°. [C. U. '51]

9. The angle of elevation of the top of a pillar is 30°, and on approaching 20 ft. nearer it is 60°. Find the height of the pillar. [G. U. '48]

10. A tree on the bank of a river is 45 ft. high and the angle of elevation of the top of the tree from a point just on the opposite bank is found to be 60°. Find the breadth of the river.

11. In advancing 60 ft. towards a tower, the altitude of its top is changed from 45° to 60°. Find the height of the tower. [E. B. S. B. '48]

12. The shadow of a tower standing on a level plane is found to be 40 ft. longer when the sun's altitude is 45° than when it is 60°. Find the height of the tower.

[C. U. '50 ; D. B. '51]

13. The horizontal distance between two pillars is 60 ft. and the angular depression of the top of the first as seen from the top of the second, which is 200 ft. high, is 30° ; find the height of the first.

14. A vertical post 120 feet high is broken at a certain height, and its upper part, not completely separated, meets the ground at an angle of 30°. Find the height at which the post is broken. [C. U. 50]

15. The upper part of a tree, broken at a certain height, makes an angle of 60° with the ground at a distance of 10 feet from its root. Find the original height of the tree.

16. The angles of elevation of the top of a monument is 30° and 60° respectively when observed from two points P and Q lying on a horizontal line through the foot of the monument and on the same side of it. If $PQ = 80$ ft., find the height of the monument. [C. U. '45 ; G. U. '52 ; E. B. S. B. '50]

17. The angle of elevation of the top of an unfinished pillar at a point 150 ft. from its base is 30° ; how much higher must the pillar be raised so that its angle of elevation at the same point may be 45° ?

18. Find the angle of elevation of the sun when the length of the shadow of a post is to its height as $\sqrt{3} : 1$.

19. A man, standing on the bank of a river, finds that the angle subtended by a post just on the opposite bank is 60°, but when he retires 60 ft. from the bank he finds the angle to be 30°. Find the height of the post and the breadth of the river.

20. From a cliff 300 ft. high the angles of depression of the top and foot of a tower are observed to be 30° and 60° respectively. Find the height of the tower.

21. From an aeroplane vertically above a straight road the angles of depression of two consecutive mile-stones are observed to be 30° and 60° respectively. Find the height of the aeroplane above the road.

22. A man stands at a point A on the bank AB of a straight river, and observes that the line joining A to a point C on the opposite bank makes with AB an angle of 30°. He then goes 200 yards along the bank to B and finds that BC makes an angle of 60° with AB. Find the breadth of the river. [C. U. '44]

23. Two posts of the same height stand on either side of a road which is 40 ft. wide. At a point on the road between the posts the elevations of their tops are 60° and 30°. Find their height and the position of the point.

24. From a point due east the angle of elevation of the top of a tower is 60° and from another point due north it is observed to be 45°. If the distance between the two points is 200 yds., fiind the height of the tower.

25. The shadow of a tower standing on a level plane is 60 ft. longer when the altitude of the sun is 30° than when it is 45°. Find the height of the tower. [W. B. S. F. '52]

26. From the top of a tree the angle of depression of an object on the hrizontal ground is found to be 60°. On descending 20 ft. from the top of the tree the angle of depression of the object is found to be 30°. Find the height of the tree. [U. U. '52]

ANSWERS

ALGEBRA

Exercise 1

1. (i) $-32a^{15}b^{10}$ (ii) $729x^{12}y^{18}$ (iii) $-\dfrac{a^5b^{10}}{32x^{10}y^5}$

(iv) $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

2. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.

3. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$.

4. $1+8a+24a^2+32a^3+16a^4$.

5. $a^5-10a^4+40a^3-80a^2+80a-32$.

6. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$.

7. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

8. $32a^5-80a^4+80a^3-40a^2+10a-1$.

9. $729-1458c+1215c^2-540c^3+135c^4-18c^5+c^6$.

10. $32a^5+80a^4b+80a^3b^2+40a^2b^3+10ab^4+b^5$.

11. $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$.

12. $243a^5+810a^4b+1080a^3b^2+720a^2b^3+240ab^4+32b^5$.

13. $a^8+8a^7+28a^6+56a^5+70a^4+56a^3+28a^2+8a+1$.

14. $x^9-9x^8+36x^7-84x^6+126x^5-126x^4+84x^3-36x^2+9x-1$.

15. $a^{12}+6a^{10}+15a^8+20a^6+15a^4+6a^2+1$.

16. $x^{10}+5x^8+10x^6+10x^4+5x^2+1$.

17. $a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8$.

18. $1-6a^2+15a^4-20a^6+15a^8-6a^{10}+a^{12}$.

19. $x^3-y^3+z^3-3x^2y+3xy^2+3x^2z+3xz^2-3yz^2+3y^2z-6xyz$.

20. $a^4+4a^3b+6a^2b^2+4ab^3+b^4-8a^3-24a^2b-24ab^2-8b^3+24a^2+48ab+24b^2-32a-32b+16$.

21. 64   **22.** 32.   **23.** 128.   **24.** 256.

24. (*a*) -1 (*b*) 0   **25.** 32.   **26.** 242.   **27.** 0.

28. 0.   **29.** 243.   **30.** 1.   **31.** 8.

32. $10x^4y+20x^2y^3+2y^5$. **33.** $12x+40x^3+12x^5$.
34. $2a^7+42a^5b^2+70a^3b^4+14ab^6$.
35. $12x^5a+40x^3a^3+12xa^5$.

Exercise 2

1. $(m^2+mn+n^2)(m^2-mn+n^2)$
2. $(x^2+2x+2)(x^2-2x+2)$ **3.** $(x^2+4x+8)(x^2-4x+8)$
4. $(2x^2+6x+9)(2x^2-6x+9)$ **5.** $(x^2+2x+3)(x^2-2x+3)$
6. $(x^2+2x-2)(x^2-2x-2)$ **7.** $(2x+z)(2x-2y-z)$
8. $(a+b-c-d)(a-b+c-d)$
9. $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$
10. $(x+y-4a)(x-y-2a)$ **11.** $(c+d+a-b)(c+d-a+b)$
12. $(a+d+b+c)(a+d-b-c)(b-c+a-d)(b-c-a+d)$
13. $(a^2b^2+ab-c+1)(a^2b^2-ab+c+1)$
14. $(x+3)(x^2-3x+9)(x-3)(x^2+3x+9)$
15. $(x-y+2)(x^2+xy-2x+y^2-4y+4)$
16. $(2a+1)(a^2+a+1)$ **17.** $3(a+b)(b-c)(a+2b-c)$
18. $(x-y+z)(x^2+y^2+z^2+xy-xz+yz)$
19. $(a+b-2c)(a^2+b^2+4c^2-ab+2ac+2bc)$
20. $(2a+b-3)(4a^2+b^2+9-2ab+6a+3b)$
21. $(a^2+2a-4)(a^4-2a^3+8a^2+8a+16)$
22. $(x^2+2x+5)(x^4-2x^3-x^2-10x+25)$
23. $(a^2-a+2)(a^4+a^3-a^2+2a+4)$
24. $3(2x-3y)(3y-z)(z-2x)$ **25.** $3(b+c)(c+a)(a+b)$
26. $(3x+11)(4x+7)$ **27.** $(3x+7y)(7x-3y)$
28. $(2a^2+5)(2a+3)(2a-3)$ **29.** $(x+a+3)(x-a-1)$
30. $(5a-3b)(15b-13a)$ **31.** $(x+1)(x-5)(x-2)^2$
32. $(a-1)(a^2-a-4)$ **33.** $(x+2)(x-1)(x+15)$
34. $(x+2)(x+3)(x+4)$ **35.** $(a^2+b^2)(x^2+y^2)$
36. $2(a+c)(1+a)(1-c)$ **37.** $(x-1)(x+1)(x-3)$
38. $(a-5)(a+2)(a^2-3a+12)$ **39.** $(x^2-3x-5)(x^2-3x-17)$
40. $(x-3)(2x+3)(2x^2-3x+7)$ **41.** $(x^2+5x+3)(x^2+5x+7)$
42. $(x+y)(y+z)(z+x)$ **43.** $(a-b)(b-c)(c-a)(a+b+c)$
44. $-(a-b)(b-c)(c-a)(ab+bc+ca)$

45. $(a+b)(b+c)(c+a)$ **46.** $(x+5)(5x+1)(5x^2+14x+20)$

47. $(a^2+3a-2)(a^2+3a-3)$ **48.** $(x^2+1)(x^2+x+1)$

49. $(x-b)(x-c)(b-c)$ **50.** $(3a-2b+c)(2a+b+3c)$

51. $(a^2-ab+b^2)(a^2-4ab+b^2)$

52. $(x+1)(x^2-3x+1)(x^2+6x+1)$

53. $-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$

54. $(x+y)(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^7)$

55. $-(x-y)(y-z)(z-x)(x+y)(y+z)(z+x)$

56. $5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$.

Exercise 3

1. (i) 20 (ii) -4 (iii) -90 (iv) $2n^3-3n^2+5n+6$.

2. (i) 40 (ii) -31 (iii) -44 (iv) 688 (v) $-3\frac{1}{4}$.

6. 3 **7.** 174 **9.** 5 **15.** $x^7+x^6y+x^5y^2+\ldots+y^7$.

16. If $p+q+r+\ldots+l+m=0$

17. $(p-p')(p'q-pq')=(q-q')^2$ **20.** ap^3+bp^2+cp+d.

21. $m=2kn$ (k being a positive integer).

Exercise 4

The values of x, y, z are given in order :

1. 13, 6 ; **2.** 5, 2 ; **3.** 6, 2 ; **4.** -3, 3, 1 ; **5.** (*a*) 4, 10 ;
5(*b*). 1, 2 ; **6.** 3, 1 ; **7.** 0, 2, 4 ; **8.** 5, 1 ; **9.** $-\frac{1}{2}$, $\frac{1}{7}$; **10.** $\frac{1}{4}$, -4 ;
11. 3, 2 ; **12.** 3, 3, 3 ; **13.** 1, 3, 5 ; **14.** 20, 15, 12 ;
15. ±4, ±6, ±7 ; **16.** $\frac{1}{2a}$, $\frac{1}{8a}$, $\frac{1}{4a}$; **16** (*a*) 2, -1, 3 ;
17. 1, 1, 1 ; **18.** a, b, c ; **19.** $bc(b-c)$, $ca(c-a)$, $ab(a-b)$;
20. 4, 3, 5 ; **21.** 1, $\frac{1}{2}$, $\frac{1}{3}$; **22.** 1, $\frac{2}{3}$, $\frac{1}{2}$;
23. $bc+ab-ca$, $ab+ca-bc$, $ca+bc-ab$; **24.** ±1, ±2, ±3 ;
25. $\frac{a}{a^2+b^2}$, $\frac{b}{a^2+b^2}$; **26.** 1, $\frac{1}{2}$, $\frac{1}{3}$; **27.** $\frac{ac-b-bc}{a^2-b^2}$, $\frac{a+ac-bc}{a^2-b^2}$;
28. $\frac{a}{2}$, $\frac{b}{2}$, $\frac{c}{2}$; **29.** 1, 2, 3 ; **30.** 4, 5, 6 ; **31.** 12, 12, 12 ;

32. 10, 20, 5 ; **33.** $-2, 2, 1$; **34.** $\frac{2}{a+b-c}, \frac{2}{a-b+c}, \frac{2}{b+c-a}$;

35. $pq+pm+qm, pqm, p+q+m$.

Exercise 5

1. Father's age 38 yrs., son's 14 yrs. ; **2.** 35, 45 ; **3.** 69 ;
4. 1215, 15 ; **5.** $\frac{15}{26}$; **6.** 93 : **7.** 72 ; **8.** 54 ;
9. 32$\frac{4}{13}$ mins. past 5 o'clock ; **10.** 84 d. ; **11.** 2 miles per hr. ;
12. 2 : 3 ; **13.** 21 tolas ; **14.** 10 hrs. ; **15.** 28 ;
16. 2 miles and 4 miles per hr. ; **17.** 4 miles and 1m. per hr. ;
18. 84 ; **19.** 1800 A. D. ; **20.** $\frac{3}{4}$ d., 512 ;
21. tea 2 s., coffee 1$\frac{1}{2}$ s. ; **22.** 123 ; **23.** 180 miles ;
24. $\frac{4}{15}$; **25.** 153 oranges, 192 apples ; **26.** 210 miles ;
27. 709 ; **28.** 27 in. ; **30.** $\frac{2}{5}$; **31.** Rs. 300, Rs. 180 ;
32. 21 miles, 3m. p.h. ; **33.** Rs. 500 ; **34.** 24, 5 ; **35.** 45 mins. ;
36. 20 m. and 10 miles per hr. ; **37.** 4 mins., 5 mins.
38. 16$\frac{2}{5}$ mins. 39. 17 mins. 52 secs. **42.** 73.

Exercise 6

1. 9 **2.** $\frac{1}{27}$ **3.** 243 **4.** 16 **5.** $\frac{1}{9}$ **6.** 125

7. $\frac{y^{\frac{1}{4}}}{x}$ **8.** x **9.** $\frac{2}{3}a^{\frac{1}{4}}b^{\frac{5}{3}}$ **10.** 1 **11.** $\left(\frac{a}{b^2}\right)^{\frac{1}{3}}$

12. -17 **13.** $a^{-\frac{17}{6}}b^{-\frac{31}{12}}$ **14.** $a^{-2}b$ **15.** $a^{\frac{32}{15x}}$ **15.** $\sqrt[3]{a}$

17. $\frac{y^5}{x^{13}}$ 17(a) $\sqrt[48]{2}$ **18.** (i) $x^{\frac{3}{4}}-x^{\frac{1}{2}}+2x^{\frac{1}{4}}-1+2x^{-\frac{1}{4}}-x^{-\frac{1}{2}}+x^{-\frac{3}{4}}$

and $1-y^{\frac{1}{3}}+x^{\frac{1}{2}}y^{-\frac{1}{3}}+y^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-1}+x^{-\frac{1}{2}}y^{\frac{1}{2}}-yx^{-\frac{1}{2}}$.

18. (ii) $a+2b^{\frac{1}{2}}+3c^{\frac{1}{3}}$ **19.** $x^{2n}-y^{-2n}$ **20.** $a^{-3m}+b^{3n}$

21. $x^{2\cdot 3^{n-1}}+x^{3^{n-1}}y^{3^{n-1}}+y^{2\cdot 3^{n-1}}$

22. $\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)$, or, $\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right)$

23. $(x^m+y^n)(x^m-y^n)$ **24.** x^a **25.** $(a^4-b^4)^m$

26. $a^{\frac{2}{n-1}}$ **27.** x^{2abc} **28.** $(x)^{-b^2}$ **29.** $\frac{1}{(x)^{p^2+q^2}}$

30. 1 **31.** 27 **32.** 1 **32** (*a*) 1 **33.** 1

34. $\left(\frac{p}{q}\right)^{m+n}$ **35.** 1 **36.** 1 **37.** $\frac{2}{3}$ **38.** $\frac{e^x-1}{e^x+1}$

39. 1 **40.** 1 **41.** $\left(\frac{a}{b}\right)^{mn}$ **42.** $\frac{x^2}{(1-x)^{\frac{3}{2}}}$

42. (*a*). 1 **43.** 1 **56.** $p=(q)^{\frac{2}{2q-1}}$.

Exercise 7

1. $\sqrt[12]{3^4}$, $\sqrt[12]{5^3}$ **2.** (i) $\sqrt{4}$, $\sqrt{\frac{9}{4}}$, $\sqrt{x^4}$ (iii) $\sqrt[3]{8}$, $\sqrt[3]{\frac{27}{8}}$, $\sqrt[3]{x^6}$
3. $\sqrt[3]{5}$ (i) $\sqrt{5}$, $\sqrt[4]{12}$, $\sqrt[3]{4}$ (ii) already arranged
(iii) $\sqrt[3]{8}$, $\sqrt[4]{9}$, $\sqrt[6]{25}$ **5.** $\sqrt{20}$, $\sqrt[3]{54}$, $\sqrt[4]{x^4y}$ **6.** (i) $5\sqrt{7}$
(ii) $8\sqrt{7}$ (iii) $40\sqrt{3}$ (iv) $5\sqrt[3]{3}$ (v) $2\sqrt[3]{3}$
9. (i) 3 (ii) $12\sqrt{5}$ (iii) 16 (iv) $6\sqrt[6]{1944}$
10. (i) $\frac{3}{2}\sqrt{10}$ (ii) $2\sqrt[3]{2}$ (iii) $\frac{1}{4}\sqrt[4]{20}$ **11.** (i) 1·06
(ii) 2·31 (iii) 6·71 (iv) 3·15 (v) 5·83
12. (i) -1 (ii) $3+\sqrt{6}+\sqrt{14}+\sqrt{21}$
(iii) $24+8\sqrt{6}+2\sqrt{15}+3\sqrt{10}$ (iv) $6\sqrt{15}-12\sqrt{2}-15+2\sqrt{35}$
(v) $2x-y$ (vi) $\sqrt{a^2+ab}-b\sqrt{a+b}-a+b^2$ (vii) $4+6\sqrt{2}$
(viii) $x\sqrt{x}+\sqrt{xy}-xy-y\sqrt{y}$ **13.** (i) $5-2\sqrt{6}$
(ii) $7-4\sqrt{3}$ (iii) $14+4\sqrt{6}$ (iv) $2x-2\sqrt{x^2-y^2}$
(v) $b^2-2a\sqrt{b^2-a^2}$ (vi) $4x+2\sqrt{4x^2-25}$ **14.** (*a*). $2+\sqrt{3}$
(*b*) $\frac{6\sqrt{10}+2\sqrt{5}+3\sqrt{2}+1}{19}$ (*c*) $5+2\sqrt{6}$
(*d*) $a+x+\sqrt{ax+x^2}$
15. $\sqrt{3}$ **16.** (*a*) 1·691 (*b*) 7·26 **17.** $2x$ **18.** $4x\sqrt{x^2-1}$
19. $\sqrt{2}$ **20.** $\frac{6(7+\sqrt{5})}{11}$ **21.** $\sqrt{a+x}$ **22.** x
23. $\frac{4}{3}$ **23** (*a*). 15 **24.** $\frac{1}{2}(3\sqrt{3}+1)$ **27.** 0 **28.** 0.

Exercise 8

[Only the positive sq. roots are given here. Students should give both the roots in each case, such as $\pm(a+b)$, $\pm(x^2-2xy+7y^2)$ etc.]

1. $a+b$ **2.** $x^2-2xy+7y^2$ **3.** x^3+3x^2-4x+2

4. $x^2-ax+2a^2$ **5.** $7x^2-5xy+6y^2$ **6,** x^2-3x+a

7. $2x^2-3xy+4y^2$ **8.** x^2-ax+b^2 **9.** $a^{-2}+a^{-1}b^{-1}-b^{-2}$

10. $2a^2+1+3a^{-2}$ **11.** $x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1$ **12.** $2x^{\frac{3}{4}}-3+4x^{-\frac{3}{4}}$

13. $2x^{\frac{3}{2}}-1+x^{-\frac{3}{2}}$ **14.** $3x+x^{\frac{1}{3}}-2x^{-\frac{1}{3}}$ **15.** $x^2-\frac{x}{2}-1$

16. $1+a+\dfrac{a^2}{2}$ **17.** $x^2-x+\frac{1}{4}$ **18.** $\dfrac{x^2}{2}-2x+\dfrac{a}{3}$

19. $x-3-\dfrac{2}{x}$ **20.** $\dfrac{x}{y}-1+\dfrac{y}{x}$ **21.** $x^2-1+\dfrac{1}{x^2}$

22. $x^2+3x+\dfrac{1}{x}$ **23.** $x^2+3x+1+\dfrac{1}{x}$ **24.** $x^2+\dfrac{2}{x}+\dfrac{1}{x^2}$

25. $\dfrac{x}{y}-\dfrac{1}{2}-\dfrac{y}{x}$ **26.** $x-\dfrac{1}{x}+1$ **27.** $a+\dfrac{1}{a}-1$

28. $2a^2+3a+\dfrac{3}{a}$ **29.** $x^3-2x+\dfrac{1}{x^3}$ **30.** $x^2+\dfrac{1}{x^2}-2$

31. $x^2+7x+11$ **32.** $x^2-11x+19$ **33.** $3x^2+7x-6$

34. $\sqrt{2}(x^2+xy+y^2)$ **35.** $\dfrac{2x}{x^2-1}$ **36.** $3-\sqrt{5}$

37. $\sqrt{3}-\sqrt{2}$ **38.** $\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}$ **39.** $2\sqrt{7}-\sqrt{5}$

40. $a-\sqrt{ax-a^2}$ **41.** $\sqrt[4]{3}(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})$ **42.** (i) $\sqrt[4]{7}(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})$

(ii) $\sqrt[4]{3}(\sqrt{3}-1)$ **43.** $\sqrt{x+3y}+\sqrt{x-3y}$

44. $2\sqrt{3x}-\sqrt{2y}$ **46.** $x=10$ **47.** $x=3$

48. $c=3$ **49.** -41 **50.** $a+2$ **51.** 10 **52.** $-a+1$

53. 1 **54.** $p^2=4q$ **55.** $n^2=4mp$

58. $1+\dfrac{a^2}{2}-\dfrac{a^4}{8}$ **59.** $a=\pm2$ **60.** $\dfrac{a^2}{b^2}-\dfrac{a}{b}+1-\dfrac{b}{a}+\dfrac{b^2}{a^2}$.

Exercise 9

1. $5a^3b^4$ **2.** $-9x^2y^{-1}$ **3.** $-\dfrac{6a^5}{7b^6}$ **4.** $2x+y$ **5.** $a-1$

6. $1+a+a^2$ **7.** $2ax-3by$ **8.** $\dfrac{2}{x^2}-3x$ **9.** $\dfrac{a}{2}+1$ **10.** 85

11. $4x^2-5y^2$ **12.** $2a+b+c$ **13.** $2a^2+a-3$ **14.** $\frac{1}{2}a-\frac{1}{3}b$

15. $\dfrac{a}{b}-1-\dfrac{b}{a}$ **16.** $x-4+\dfrac{2}{x}$ **17.** 2.

Exercise 10

1. ± 5 **2.** ± 1 **3.** $\pm\sqrt{2}$ **4.** ± 2

5. $\frac{4}{3}, -\frac{5}{14}$ **6.** $3, \frac{1}{3}$ **7.** $14, 2\frac{1}{4}$ **8.** $\frac{5}{2}, -\frac{2}{3}$

9. $7, -4\frac{9}{17}$ **10.** $9\frac{15}{17}, -11$ **11.** $5\frac{2}{3}, 9\frac{1}{2}$ **12.** $3, 23$

13. $2, -\frac{9}{2}$ **14.** $3\pm\sqrt{7}$ **15.** $5\pm\sqrt{17}$ **16.** $c, c-2b$

17. $\dfrac{b\pm\sqrt{b^2+4ac}}{2a}$ **18.** $6, \frac{1}{6}$ **19.** $12, \frac{3}{4}$

20. $-3, 2$ **21.** $3, -4$ **22.** $6, 9$ **23.** $4, -2\frac{1}{4}$

24. $3, -\frac{68}{9}$ **25.** $9, \frac{15}{13}$ **26.** $-4, 2$ **27.** $5, \frac{6}{5}$

28. $0, -7$ **29.** $0, 2\frac{1}{4}$ **30.** $-2a, -3a$

31. $\dfrac{3\pm\sqrt{10}}{2}$ **32.** $-a, -b$ **33.** $0, a+b$ **34.** $\sqrt{13}\pm 3$

35. $\sqrt{3}\pm 4$ **36.** 7 **37.** 6 **38.** $\frac{2}{5}$ **39.** 8 (extraneons root $\frac{1}{5}$)

40. $\dfrac{3\pm\sqrt{61}}{2}$ **41.** $\dfrac{a(m-1)^2}{2m-1}$ **42.** ± 6 **43.** 25 **44.** 5

45. 25 **46.** $\dfrac{ab}{a+b}$ **47.** $-64, 27$ **48.** $1, \dfrac{1}{\sqrt[3]{4}}$ **49.** $\frac{1}{2}$

50. 3 **51.** 5 **52.** $2, 1$ **53.** $\frac{1}{2}$ **54.** 1 **55.** 4 **56.** $2, 3$

57. $0, -1$ **58.** $2, 1$ **59.** $-2, -2$ **60.** ± 2 **61.** $2, 3$

62. $x=1, y=\frac{1}{2}$ **63.** $2, -1$ **64.** $0, 3$ **65.** $-\frac{37}{29}$

66. $x=5, y=3$ **67.** $1, -2$ **68.** $x=3, y=3$

69. $\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{2}{3}}$ **70.** $\frac{2}{3}, -4$ **71.** $\pm\dfrac{2p}{\sqrt{5}}$

72. $-\frac{2}{3}(1 \pm \sqrt{7})$ **73.** 43, −42 **74.** $2\frac{3}{7}, -1\frac{4}{9}$
75. $1, \frac{1}{a}$ **76.** 1·85, −1·12.

Exercise 11

1. 4 or −1 or −4 **2.** $\frac{2}{3}, \frac{4}{3}$ **3.** 5, 7 or −7, −5
4. 10, 11 or −11, −10 **5.** 7 or −6 **6.** 11,13, or −13, −11
7. 121, 100 (in the second case −10 is taken as sq. root)
8. 6, 8, or −8, −6 **9.** 8″, 15″ **10.** 5″, 12″
11. A 2 hrs., B 3 hrs. **12.** 9 d. **13.** 20 **14.** 76
15. 6 miles per hr. **16.** 8″ **17.** 20′, 13′
18. 4 miles per hr. **19.** 19, 20 **20.** 7″
21. 8 m., 12 m. **22.** 324.

Exercise 12

1. (*a*) $x=3, y=1$ (*b*) $x=5, y=1$ (*c*) $x=-1$ (*d*) $x=1$ (*e*) $x=7$
21. (5, 12), (12, 5) **22.** (1, 1) **23.** (0, 0), (4, 4)
25. (−2, 1), (4, 4) **26.** 1·4 **27.** (3, 4), (4, 3)
28. 2·3 **29.** (5, 2) **30.** 1 **31.** −1, 2
32. 1, 1·5 **34.** 3, −1 **35.** 6·7 units (app.) **36.** 1, 3
37. 2·4, −·4. **38.** 7·5 units **39.** 1 **40.** 3, ·5
41. −1, −2 **43.** 2, 4 **44.** (0, 0), (1, 1) **45.** 4, −1
46. (i) 2 ; (ii) −1 ; (iii) ·4, −2·4 **52.** (−3, 4) **53.** $5, -\frac{1}{2}$
55. $10\frac{1}{2}$ annas, (2) 14 ; **56.** At 5 p. m., at a distance of 35 mi.
57. (1) 50 mins. after P starts, 25 miles from Howrah ; (2) 6 miles.
58. 4 hrs. 12 mins. after A starts, 13·2 miles from starting place ;
59. At 9-9 A.M., 26 miles away from Howrah ; **60.** 80 miles ;
61. 2·4 hrs. ; **62.** (*a*) at 5-24 P.M., 24 m. from P ; (*b*) 4 miles ;
63. (i) Rs. 80, (ii) Rs. 240 per month ; **64.** 24,500
65. 3·4 Kg., 15·4 lb. **67.** $x=4, y=5$, or $x=-1{\cdot}6, y=-6{\cdot}2$
68. 4, −8 **69.** $x=12, y=2$; $x=-4, y=-6$.
70. $x=3, y=8$; $x=8, y=3$; $x=-3, y=-8$: $x=-8, y=-3$
71. $(\frac{9}{5}, \frac{16}{5})$. **72.** (5, 12), (−5, 12) **73.** $x=1, y=4$ and
$x=-4, y=-1$ **74.** 2 **75.** (4, 6).

TRIGONOMETRY

Exercise 1

1. (i) $3^\circ\,40'\,14''$ (ii) $8^\circ\,27'\,30''$ (iii) $21^\circ\,36'$
2. $4^g\,62^`\,35^{``}$ (ii) $7^g3^`33\cdot\dot{3}^{``}$ (iii) $10^g\,72^`\,5^{``}$ (iv) $3^g\,24^`\,60^{``}$
3. (i) $\frac{1}{8}$ (ii) $\frac{27}{40}$ (iii) $1\cdot35$ (iv) $4\frac{13}{3375}$ (v) $\cdot503524$ (vi) $\frac{2}{8}$ (vii) 20
4. (i) $33^g\,56^`\,66\cdot\dot{6}^{``}$ (ii) $70^g\,27^`\,50^{``}$ (iii) $66^g79^`\,16\cdot\dot{6}^{``}$ (iv) 120^g
(v) $16^g\,66^`\,66\cdot\dot{6}^{``}$
5. (i) $36^\circ\,19'\,1\cdot776''$ (ii) $54^\circ 8\cdot1''$ (iii) $36^\circ\,24'\,29\cdot664''$
(iv) $114^\circ\,32'\,43\cdot\dot{6}\dot{8}''$ (v) 100° (vi) 120°
6. (i) $\frac{\pi}{6}$ (ii) $\frac{151\pi}{360}$ (iii) $\frac{3407\pi}{13500}$ (iv) $\frac{3}{5}\pi$ (v) $\frac{44}{125}\pi$
(vi) $1\cdot626286\pi$ **7.** $35^\circ, 19^\circ$; $\frac{350}{9}g, \frac{190}{9}g$
8. $110\cdot5^\circ$, $29\cdot5^\circ$, $\frac{221\pi^c}{360}$, $\frac{59\pi^c}{360}$
9. (i) $100^\circ, 60^\circ$ (ii) $250^g, 150^g$; (iii) $30\pi^c, 18\pi^c$ **10.** $66\cdot\dot{6}^g$
11. $\frac{4\pi}{9}, \frac{2}{5}\pi$ **11.** (*a*) $40^g, 60^g, 90^g$ **12.** $22\frac{1}{2}^\circ, 67\frac{1}{2}^\circ$ **13.** $\frac{\pi}{4}$
14. 63° **15.** $66^\circ40', 21^\circ36'$ **16.** $\frac{\pi}{3}$ **17.** 96°
18. $\frac{(n-2)\pi}{n}$ **19.** $\left(\frac{90}{\pi}+\frac{1}{2}\right)^\circ, \left(\frac{90}{\pi}-\frac{1}{2}\right)^\circ$ or $\frac{641^\circ}{22}, \frac{619^\circ}{22}$
21. $54^\circ, 81^\circ, 108^\circ, 135^\circ, 162^\circ$; $\frac{3\pi}{10}, \frac{9\pi}{20}, \frac{3\pi}{5}, \frac{3\pi}{4}, \frac{9\pi}{10}$
22. 28 yds. **23.** 90° **24.** $6\frac{32}{35}$ in. **25.** $3\frac{1}{7}$ in.
26. 2500 ft. **27.** 110 in. **28.** 856720 miles (App.)
29. $55\frac{1}{2}$ ft. **30.** $\frac{55}{567}$ miles **31.** $1^\circ25'\,57''$ (App.)
32. 1 mi. 287 yds. 3 ft. (App.) **33.** $\frac{4\pi}{35}, \frac{9\pi}{35}, \frac{14\pi}{35}, \frac{19\pi}{35}, \frac{24\pi}{35}$
34. $123\cdot899$ sq. ft. **35.** 200 ft. **36.** $94\frac{1}{4}''$ (App.)
37. At 5-12 P.M., or at $42\frac{6}{11}$ min. past 5 **38.** 42, 14
39. $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$ **41.** at 1-36 P.M. **42.** $7'9\cdot\dot{5}\dot{4}''$
43. 5 : 4 **44.** $\frac{4\pi^c}{5}$.

Exercise 2

33. 2 **34.** 0 **35.** 1 **36.** 0 **37.** 0 **38.** 0.

Exercise 3

1. $\sin\theta=\sqrt{1-\cos^2\theta}$, $\tan\theta=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$, $\cot\theta=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$,
$\sec\theta=\frac{1}{\cos\theta}$, $\operatorname{cosec}\theta\frac{1}{\sqrt{1-\cos^2\theta}}$ **2.** $\frac{3}{4}, \frac{5}{4}$ **3.** $\frac{12}{5}, \frac{13}{5}$
4. $\frac{3}{5}, \frac{5}{4}$ **5.** $\frac{3}{4}$ **6.** $\frac{60}{11}, \frac{61}{60}$ **13.** $\sin=\frac{4}{5}$, $\cos=\frac{3}{5}$, $\cot=\frac{3}{4}$,
$\sec=\frac{5}{3}$, $\operatorname{cosec}=\frac{5}{4}$ **14.** $\frac{4}{5}$ **14.** $\frac{1}{7}$ **16.** $\frac{16}{63}$.

Exercise 4

2. $\frac{1}{2}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ **3.** 9 **4.** 1 **5.** $1\frac{1}{4}$
6. $9\frac{2}{3}$ **11.** 0 **12.** $\frac{5\sqrt{3}}{6}$ **13.** 1.

Exercise 5

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **2.** $x^2+y^2=b^2$ **3.** $xy=a^2$
4. $ac=b^2$ **5.** $xy=ab$.

Exercise 6

1. 30° **2.** 60° **3.** $\theta=30°$ or 60° **4.** 0° or 90°
5. 60° **6.** 45° **7.** 45° **8.** 6°
9. 5° **10.** $\frac{1}{\sqrt{2}}$ **11.** $x=3$, $\theta=30°$ **12.** $p=2$, $\theta=60°$
13. $-\frac{5}{3}$ **14.** $\frac{\sqrt{3}}{2}$ 17. 60° **18.** 0°, 60° **19.** 30°.

Exercise 7

1. 173·2 ft. or $100\sqrt{3}$ ft. **2.** $150\sqrt{3}$ yds. **3.** 103·92 ft.
4. 10·392 ft. **5.** 9 ft. **6.** $15\sqrt{2}$ ft. **7.** 30°
8. 68·3 ft. **9.** 17·32 ft. **10.** $15\sqrt{3}$ ft.
11. 141·96 ft. **12.** 94·64 ft. **13.** 165·36 ft. (app.) **14.** 40 ft.
15. 37·32...ft. **16.** $40\sqrt{3}$ ft. **17.** 63·4 ft. (app.) **18.** 30°
19. 51·96 ft., 30 ft. **20.** 200 ft. **21.** ·866 mi., or ·433 mi.
22. 173·2 yds. **23.** height = 17·32 ft., 10 ft. from one post
24. 173·2 yds. **25.** 81·96 ft. (app.) **26.** 30 ft.

APPENDIX

SOME ADDITIONAL THEOREMS

Brahmagupta's Theorem

1. If from the vertex of a triangle a perpendicular is drawn to the base, the rectangle contained by the other two sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.

[C. U. '39 Sup.]

Let ABC be a triangle and AD be perpendicular from A to the base BC and let AE be a diameter of the circum-circle.

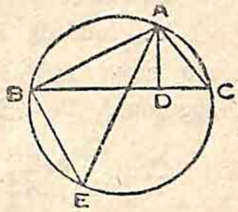


fig. 1

To prove that AB.AC = AD.AE.

Join BE.

Proof: In $\triangle^s$ABE and ADC,

$\angle$E = $\angle$C in the same segment,

$\angle$ABE in the semi-circle = 1 rt. angle = $\angle$ADC.

$\therefore$ $\triangle^s$ABE, ADC are equiangular, $\therefore$ $\frac{AB}{AD}=\frac{AE}{AC}$,

$\therefore$ AB.AC = AD.AE.

Corollary: If a, b, c denote the sides of a triangle and S its area, then R, the circum-radius of the triangle is given by $R=\frac{abc}{4S}$. [C. U. '39 Sup.]

Of the $\triangle$ABC, the area $S=\frac{1}{2}BC.AD$, $\therefore$ $4S=2BC.AD=2a.AD$.

From the above theorem, AE.AD = AB.AC, or, $2R.AD=c.b$,

$$\therefore\quad 2R=\frac{bc}{AD}=\frac{abc}{a.AD},\quad \therefore\quad R=\frac{abc}{2a.AD}=\frac{abc}{4S}.$$

Ptolemy's Theorem

2. The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by the pairs of opposite sides.

[C. U. '36, '38, '43, '44 ; D. B. '31, '44, '51 ; G. U. '49, '51]

Let AC and BD be the diagonals of the cyclic quadrilateral ABCD.

To prove that $AC.BD = AB.CD + BC.AD$.

Construction : Draw $\angle DAP = \angle BAC$, and let AP cut BD in P.

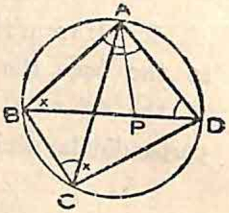


fig. 2

Proof : $\because$ $\angle BAC = \angle PAD$,

$\therefore$ adding $\angle PAC$ to both sides, we have $\angle BAP = \angle CAD$; and $\angle ABD = \angle ACD$ in the same segment. $\therefore$ Δ^s ABP, ACD are equiangular.

$$\therefore \frac{BP}{CD} = \frac{AB}{AC}, \quad \therefore BP.AC = AB.CD \cdots\cdots (1).$$

Again, in Δ^s APD and ABC, $\angle ADP = \angle ACB$ in the same segment, and $\angle PAD = \angle BAC$ (by construction),

$\therefore$ the triangles are equiangular.

$$\therefore \frac{PD}{BC} = \frac{AD}{AC}, \quad \therefore PD.AC = BC.AD \cdots\cdots (2).$$

Now, adding (1) and (2) we have

$BP.AC + PD.AC = AB.CD + BC.AD$.

But $BP.AC + PD.AC = AC(BP + PD) = AC.BD$,

$\therefore$ $AC.BD = AB.CD + BC.AD$.

Ceva's Theorem

3. If three straight lines drawn from the vertices of a triangle to meet the opposite sides are concurrent, then the product of the three ratios of the segments into which they divide the sides is equal to 1.

Or, [If three concurrent st. lines are drawn from the angular points of a triangle to meet the opposite sides or these sides produced, the product of the three alternate segments taken in order is equal to the product of the other three segments.]

[C. U. '40]

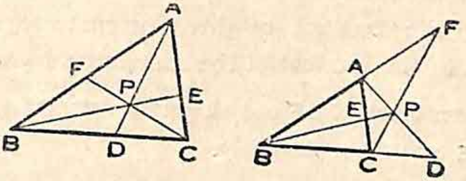


Fig. 3

In $\triangle ABC$, let the st. lines AD, BE, CF meet the opposite sides BC, AC, AB respectively at D, E, F and let them be concurrent at P.

To prove that $\frac{BD}{DC}.\frac{CE}{AE}.\frac{AF}{BF}=1.$

Proof : $\because$ $\triangle ABP$ and $\triangle BPC$ stand on the same base BP,

$\therefore$ $\frac{\triangle ABP}{\triangle BPC}=\frac{AE}{CE}$. Similarly $\frac{\triangle BPC}{\triangle APC}=\frac{BF}{AF}$

and $\frac{\triangle APC}{\triangle ABP}=\frac{CD}{BD}$.

Multiplying these ratios we have,

$$\frac{AE}{CE}.\frac{BF}{AF}.\frac{CD}{BD}=\frac{\triangle ABP}{\triangle BPC}.\frac{\triangle BPC}{\triangle APC}.\frac{\triangle APC}{\triangle ABP}=1,$$

$$\therefore \quad \frac{BD}{CD}.\frac{CE}{AE}.\frac{AF}{BF}=1.$$

[**N. B.** In answering C. U. '40 question, we have to add $\therefore$ BD.CE.AF = CD.AE.BF.]

Menelaus, Theorem

4. If a transversal cuts the sides of a triangle, then the product of the three ratios of the segments into which it divides the sides is equal to **1.**

Or, [If a straight line cuts the sides or the sides produced of a triangle, the product of three alternate segments taken in order is equal to the product of the remaining segments.]

Let the transversal DEF cut the sides BC, CA and BA of the $\triangle$ABC in D, E and F respectively.

To prove that $\frac{BD}{DC} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$

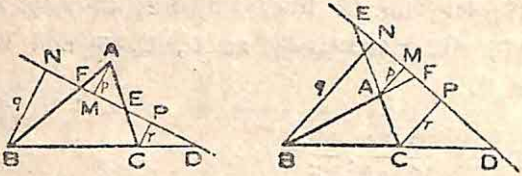


Fig. 4

Construction : Draw AM, BN and CP perpendiculars to the st. line DEF. Let AM $= p$, BN $= q$ and CP $= r$ units of length.

Proof : $\because$ CP and BN are perpendicular to DF,

$\therefore$ CP $\parallel$ BN.

Hence $\triangle$'BND and CPD are equiangular, $\therefore \frac{BD}{CD} = \frac{q}{r}$.

Similarly $\triangle$CPE and $\triangle$AME are equiangular, $\therefore \frac{CE}{AE} = \frac{r}{p}$;

and $\triangle$AMF and $\triangle$BNF are equiangular, $\therefore \frac{AF}{BF} = \frac{p}{q}$.

Multiplying these ratios we have,

$$\frac{BD}{DC} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{q}{r} \cdot \frac{r}{p} \cdot \frac{p}{q} = 1.$$

5. Perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

[C. U. '30. '37 ; D. B. '26, '49, '50]

Or, The altitudes of a triangle are concurrent.

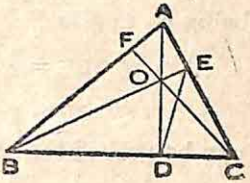


Fig. 5

Let AD and BE be perpendiculars to BC and AC respectively in △ABC and let them intersect at O. Join CO and produce it to meet AB at F.

To prove that CF is perp. to AB.

Proof : Join DE. ∵ ∠BEA and ∠BDA are right angles, ∴ ABDE is a cyclic quadrilateral.

∴ ∠BAD = ∠BED = ∠OED (in the same segment).

Again, ∠OEC and ∠ODC being rt. angles, ODCE is a cyclic quadrilateral.

∴ ∠OED = ∠OCD. ∴ ∠OAF = ∠OED = ∠OCD.

Now, ∠COD + ∠OCD = 1 rt. angle (∵ ∠ODC is a rt. ∠), and ∠COD = ∠AOF, ∴ ∠OAF + ∠AOF = ∠COD + ∠OCD = 1 rt. angle, ∴ ∠AFO is a rt. angle. ∴ CF⊥AB.

∴ The perpendiculars are concurrent.

[**N. B.** The point of intersection of the altitudes is called the **ortho-centre** of the triangle. It is denoted by O.]

6. In an acute-angled triangle, the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle.

[C. U. '49]

[**N. B.** The triangle formed by joining the feet of the altitudes of a triangle is called the **pedal triangle** with respect to the original triangle.]

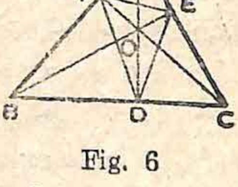


Fig. 6

Let the perpendiculars AD, BE and CF on the sides of △ABC be concurrent at O, and let DEF be the pedal triangle formed.

To prove that ∠D, ∠E and ∠F of the △DEF are bisected by AD, BE and CF respectively.

Proof : ∵ ∠OFD and ∠ODB are right angles, ∴ BDOF is a cyclic quadrilateral.

$\therefore$ $\angle ODF = \angle OBF$ in the same segment. Again, $\because$ $\angle CFB$ and $\angle BEC$ are right angles, $\therefore$ BCEF is a cyclic quad., $\therefore$ $\angle EBF = \angle FCE$, *i.e.*, $\angle OBF = \angle OCE$, $\therefore$ $\angle ODF = \angle OCE$.

Now, $\because$ ODCE is a cyclic quadrilateral ($\angle ODC$, $\angle OEC$ being rt. angles),

$\therefore$ $\angle OCE = \angle ODE$ in the same segment.

$\therefore$ $\angle ODF = \angle ODE$. $\therefore$ AD bisects the $\angle FDE$. Similarly it can be proved that BE bisects $\angle FED$ and CF bisects $\angle DFE$.

7. The distance of the circum-centre from any side of a triangle is half the distance of the ortho-centre from the opposite vertex. [cf. C. U. '10]

Let the altitudes AD, BE and CF of $\triangle ABC$ meet at O. Then O is the ortho-centre. Let S be the circum-centre of the triangle and P, Q, R be respectively the mid points of BC, CA, AB. Then SP, SQ, SR are perpendicular to BC, AC, AB respectively.

To prove that $SP = \frac{1}{2}AO$, $SQ = \frac{1}{2}BO$ and $SR = \frac{1}{2}CO$.

Let X and Y be the mid points of AO and BO respectively.

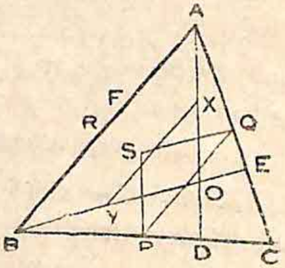


Fig. 7

Join XY and PQ.

Proof : $\because$ X, Y are the mid points of AO, BO,

$\therefore$ XY is parallel to and half of AB. Again, P and Q are the mid points of BC and AC, $\therefore$ PQ is parallel to and half of AB.

$\therefore$ PQ and XY are equal and parallel.

Now, $\because$ SP and AD are perp. to BC, $\therefore$ $SP \parallel AD$ or XO. Similarly $SQ \parallel BE$ or YO. $\therefore$ the $\angle SPQ$ between SP and $PQ = \angle YXO$ between XO and YX, and similarly $\angle PSQ = \angle XOY$.

∴ △PSQ and △XOY are congruent. ∴ SP = XO = $\frac{1}{2}$AO. Similarly it can be proved that SQ = $\frac{1}{2}$BO and SR = $\frac{1}{2}$CO.

8. The feet of the perpendiculars drawn to the three sides of a triangle from any point of its circum-circle are collinear.

[H.S. '61 (Compl.) ; C.U. '36, '39, '41, '50 ; G. U. '50, '52]

Let P be any point on the circum-circle of △ABC, and let PD, PE and PF be perpendicular to AB, BC and CA respectively.

To prove that D, E and F are collinear. Join DE, DF, PA, PB.

Proof : ∵ ∠PDA and ∠PFA are rt. angles, ∴ quad. PDAF is cyclic. ∴ ∠PDF = ∠PAF in the same segment = ∠PBC (∵ quad. APBC is cyclic).

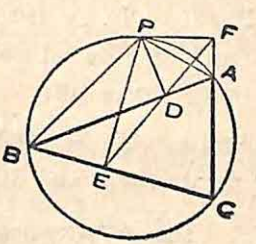


Fig. 8

Again, ∵ ∠PDB and ∠PEB are right angles,

∴ quad. PDEB is cyclic. ∴ ∠PBE + ∠PDE = 2 right angles. But ∠PDF = ∠PBE (proved). ∴ ∠PDF + ∠PDE = 2 rt. angles. ∴ DE and DF are in the same st. line.

∴ D, E and F are collinear.

[N. B. The st. line on which the feet of the above perpendiculars lie is called **Pedal line** or **Simson line** of the point P with respect to the △ABC.]

9. In any triangle the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the joins of the ortho-centre to the vertices are concyclic.

[C. U. '26, '40, '44, '50 ; W. B. S. F. '52 ; G. U. '49, '52]

Let D, E, F be the feet of the perpendiculars AD, BE and CF on the sides of the △ABC, and let P, Q, R be the mid points of BC, AC, AB respectively and X, Y, Z be respectively the mid points of AO, BO, and CO, O being the ortho-centre.

To prove that D, E, F, P, Q, R, X, Y, Z are concyclic.

Join PX, RX, RP, QX and PQ.

Proof : In $\triangle ABO$, R and X are the mid points of AB and AO, $\therefore$ RX ∥ BO or BE.

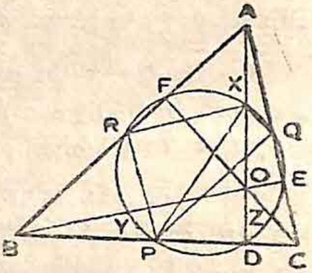


Fig. 9

Again, $\because$ R and P are the mid points of AB and BC respectively, $\therefore$ RP ∥ AC.

$\therefore$ the $\angle PRX$ between PR and RX + the $\angle AEB$ between AE and BE = 2 rt. angles, but $\angle AEB$ is a rt. angle,

$\therefore$ $\angle PRX = 1$ rt. angle. It can be similarly shown that $\angle PQX$ is a rt. angle. $\therefore$ PX subtends right angles at the pts. R, Q, D. $\therefore$ the circle drawn on PX as diameter will pass through the points P, X, R, D, Q. Similarly it can be proved that the circle passes through the pts. E, F, Y, Z. Hence the nine points D, E, F, P, Q, R, X, Y, Z are concyclic.

[N. B. (i) The circle that passes through those nine points is called the **Nine-point Circle** of the $\triangle ABC$; (ii) the centre of this circle is called the **Nine-point Centre.** (iii) To draw this circle, join the middle point of the distance between the ortho-centre and any vertex and the middle point of the opposite side, and then the circle drawn on this joining st. line as diameter is the nine-point circle.]

Miscellaneous Solutions

Ex. 1. Prove that the circum-centre, the centroid and the ortho-centre of a triangle are collinear. [C. U. '27, '51]

Let S be the circum-centre and O the ortho-centre of the $\triangle$ABC and let D be the middle point of BC. Join SD, SO and AD. Let the median AD cut SO at G.

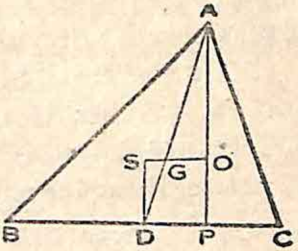


Fig. 10

To prove that G is the centroid of the triangle.

Proof : $\because$ SD and AP are perp. to the same st. line BC, $\therefore$ SD $\parallel$ AO.

$\therefore$ $\angle$GSD = alt. $\angle$AOG and $\angle$SGD = vert. opp. $\angle$AGO ;

$\therefore$ Δ^sGSD, AGO are equiangular.

$\therefore$ $\frac{GD}{AG} = \frac{SD}{AO} = \frac{1}{2}$ [$\because$ the distance of the circum-centre from any side $= \frac{1}{2}$ the distance of the ortho-centre from the opposite vertex.], $\therefore$ GD $= \frac{1}{2}$AG.

$\therefore$ G is the point of trisection of the median AD.

$\therefore$ G is the centroid of the triangle.

$\therefore$ The circum-centre S, the ortho-centre O and the centroid G of the triangle are collinear.

Ex. 2. The centre of the nine-point circle is the middle point of the straight line which joins the ortho-centre to the circum-centre.

Let S be the circum-centre and O the ortho-centre of $\triangle$ABC. To prove that the centre of the nine-point circle is the middle point of SO.

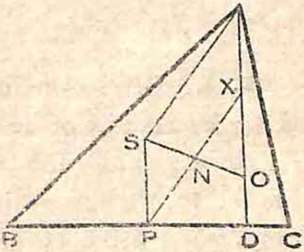


Fig. 11

Construction : Join SO. Let P and X be the middle points of BC and AO. Join PX, which cuts SO in N. Join SP. Join AO and produce it to meet BC in D.

Proof : $\because$ S is the circum-centre and P the middle pt. of BC,

∴ SP⊥BC. ∵ SP, AD are perp. to BC, ∴ SP ∥ AD. Again, SP = $\frac{1}{2}$AO = OX. Now, in Δ^sSPN and XON, SP = OX, ∠PSN = alt. ∠XON and ∠SNP = vert. opp. ∠XNO ;

∴ SN = ON and PN = NX, ∴ N is the middle point of SO and PX. ∵ PX is the diameter of the nine-point circle, ∴ its mid pt. N is the centre of the nine-point circle. ∴ the centre of the nine-point circle is the middle point of SO.

Ex. 3. Prove that the radius of the nine-point circle is half the radius of the circum-circle. [C. U. '40 ; G. U. '50]

[**Hints** : (see fig. 11) SP = $\frac{1}{2}$AO = AX and SP ∥ AD or AX, ∴ SP, AX are equal and parallel. ∴ ASPX is a parallellogram. ∴ PX = SA. ∴ $\frac{1}{2}$PX = $\frac{1}{2}$SA. Since PX is the diameter of the nine-point circle and SA is the radius of the circum-circle, ∴ the radius of the nine-point circle is half the radius of the circum-circle.]

Ex. 4. Prove that the circum-centre, the ortho-centre, the centroid and the nine-point centre of a triangle are collinear. [C. U. '34]

[**Hints** : Draw fig. 10 and prove as in Ex. 1 above, that S, G, O (*i e.*, circum-centre, centroid and ortho-centre) are in the same st. line. Now, join XD, X and D being the mid points of AO and BC. Let XD cut SO at N. Now show, as in Ex. 2 above, that N is the centre of the nine-point circle. Since N lies on SO, the points S, O, G and N are collinear.]

Locus

Ex. 1. Given the base and the vertical angle of a triangle, find the locus of its ortho-centre.

[Draw a fig.] Let ABC be any triangle, whose base BC is equal to the given base and vertical ∠A is equal to the given angle. From B and C draw BD and CE perpendiculars to the opposite sides. Let BD, CE meet at O.

To find the locus of the ortho-centre O.

$\because$ $\angle D$ and $\angle E$ of the quadrilateral AEOD are right angles, $\therefore$ $\angle DOE + \angle A = 2$ rt. angles ; but as $\angle A$ is fixed or constant, $\therefore$ $\angle BOC = \angle DOE =$ constant.

$\therefore$ BC subtends a constant angle at O.

$\therefore$ the arc BOC of the circle passing through B, O, C (*i.e.*, of the segment of a circle on BC containing $\angle BOC$) is the locus of O.

Ex. 2. Given the base and the vertical angle of a triangle, find the locus of its in-centre. [C. U. '19]

[**Hints** : Let the bisectors of $\angle B$ and $\angle C$ of the $\triangle ABC$ meet at I, which is the in-centre. To find the locus of I.

In $\triangle BIC$, $\angle BIC + \angle IBC + \angle ICB = 180^\circ$, *i.e.*,

$\angle BIC + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 180^\circ \cdots (1)$. In $\triangle ABC$, $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90^\circ \cdots (2)$. Subtracting (2) from (1), $\angle BIC - \frac{1}{2}\angle A = 90^\circ$, $\therefore$ $\angle BIC = 90^\circ + \frac{1}{2}\angle A =$ constant ($\because$ $\angle A$ is a fixed angle).

$\therefore$ BC which is constant, subtends a constant angle at I, $\therefore$ the arc BIC of the segment of a circle on BC containing $\angle BIC$ is the locus of the in-centre I.

Ex. 3. Given the base and the vertical angle of a triangle, find the locus of its centroid.

Let ABC be any triangle whose base BC is equal to the given base and vertical angle A is equal to the given angle. Let the medians BE and CF meet at G.

To find the locus of the centroid G.

Draw GP $\parallel$ AB and GK $\parallel$ AC and let GP, GK meet BC in P, K respectively.

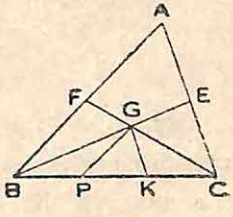


fig. 12

$\because$ GP $\parallel$ AB, $\therefore$ $\angle GPK =$ cor. $\angle ABC$, and

$\because$ GK $\parallel$ AC, $\therefore$ $\angle GKP =$ cor. $\angle ACB$,

$\therefore$ $\angle PGK = \angle A =$ constant.

Again. $\because$ the medians of a triangle meet at a point of trisection, $\therefore$ $FG = \frac{1}{3}CF$ and $EG = \frac{1}{3}BE$.

Now, since GP ∥ FB,

$\therefore \frac{BP}{BC} = \frac{FG}{FC} = \frac{1}{3}$, $\therefore$ BP $= \frac{1}{3}$BC. Similarly KC $= \frac{1}{3}$BC. $\therefore$ PK $= \frac{1}{3}$BC.

$\because$ BC is fixed, $\therefore$ PK is of fixed length and it subtents a constant angle at G. Hence the arc PGK of the segment of a circle on PK, containing ∠A, is the locus of G.

Ex. 4. Given the base and the vertical angle of a triangle, find the locus of the nine-point centre.

Let ABC be any triangle whose base BC and vertical ∠A are fixed or constant, and let N be the centre of its nine-point circle. To find the locus of N.

Let D be the mid point of BC. Join DN, then DN is the radius of the nine-point circle. $\because$ radius of the nine-point circle is half the radius of the circum-circle,

$\therefore$ DN $= \frac{1}{2}$ the circum-radius of △ABC. Again, since the base BC and the vertical ∠A of the triangle ABC are fixed or constant its circum-circle is also fixed, $\therefore$ DN is constant.

$\because$ BC is fixed, $\therefore$ its mid point D is also fixed. Thus the distance of the moving point N from the fixed point D is always constant, being equal to DN. Hence the circumference of the circle drawn with centre D and with radius DN (*i.e.*, half the circum-radius) is the locus of N.

Maxima and Minima

Ex. 1. A and B are two points on the same side of an unlimited st. line. Find a point P on the st. line such that the sum of the distances of the point from A and B is a minimum. [C. U. ; D. B. '49, '50]

Let A and B be two points on the same side of an unlimited straight line EF.

To find a pt. P on EF so that the sum of its distances from A and B is a minimum.

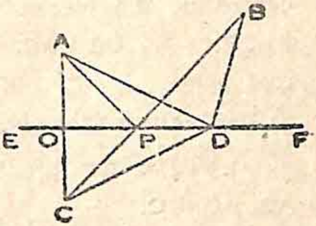


Fig. 13

Construction : Draw $AO \perp EF$ and produce AO to C, making $CO = AO$. Join CB, cutting EF at P. Then P is the required point.

Proof : Take any other point D on EF. Join AP, AD, CD, BD.

$\because$ P and D are on the perpendicular bisector of AC, $\therefore$ $AP = CP$ and $AD = CD$.

Now, $AP + BP = CP + BP = BC$; but $BC < BD + CD$,

$\therefore$ $BC < BD + AD$. $\therefore$ $AP + BP < AD + BD$, and this is true for any position of D on EF. $\therefore$ P is the required point.

[**N. B.** $AP + BP$ is the least, when AP and BP make equal angles with EF.]

Ex. 2. Of all triangles standing on the same base and having the same vertical angle, prove that the isosceles triangle has (i) the greatest area, (ii) the greatest perimeter.

[E. B. S. B. '50 ; C. U. '41]

Let ABC be an isosceles triangle and ABD be any other triangle standing on the base AB and having equal vertical angles ACB and ADB.

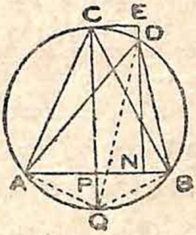


Fig. 14

To prove that (i) $\triangle ABC > \triangle ABD$, and (ii) the perimeter of $\triangle ABC >$ the perimeter of $\triangle ABD$.

(i) Draw $CP \perp AB$ and $DN \perp AB$.

Proof : $\because$ AB subtends equal angles at C and D, $\therefore$ A, B, D, C are concyclic. Draw CE tangent to the circle ABDC at C. Let ND produced meet CE in E. Now, since CP is perp. to the base AB from the vertex C of the isosceles

$\triangle ABC$, $\therefore$ CP bisects AB. $\therefore$ CP passes through the centre of the circle. $\therefore$ The tangent CE is perpendicular to CP. $\therefore$ CE $\parallel$ AB. Again, $\because$ CP $\parallel$ EN, $\therefore$ CPNE is a rectangle. $\therefore$ $CP=EN$. $\because$ $CP>DN$. $\therefore$ the height of $\triangle ABC>$ the height of $\triangle ABD$. $\because$ $\triangle ABC>\triangle ABD$.

(ii) Let CP produced meet the circumference in Q. Join AQ, BQ, DQ.

Proof : $\because$ Q is on the perp. bisector of AB, $\therefore$ $AQ=BQ$. $\because$ the chord CQ passes through the centre, $\therefore$ CQ is a diameter. $\therefore$ $CQ>$ the chord DQ.

Now, $\because$ ACBQ is a cyclic quadrilateral,

$\therefore$ $BQ.AC+AQ.BC=AB.CQ$,

or, $AQ(AC+BC)=AB.CQ$ [$\because$ $AQ=BQ$]…(1)

Again, $BQ.AD+AQ.BD=DQ.AB$,

or, $AQ(AD+BD)=DQ.AB$…(2)

$\therefore$ from (1) and (2), we have $\dfrac{AQ(AC+BC)}{AQ(AD+BD)}=\dfrac{AB.CQ}{AB.DQ}$,

or, $\dfrac{AC+BC}{AD+BD}=\dfrac{CQ}{DQ}$; but $CQ>DQ$, $\therefore$ $AC+BC>AD+BD$.

$\therefore$ $AC+BC+AB>AD+BD+AB$.

Ex. 3. Of the st. lines through a point of intersection of two circles and terminated by the circumferences, the maximum is that which is parallel to the line of centres.

Let P and Q be the centres of the circles intersecting at A and B. Through A draw CD $\parallel$ PQ, so that CD cuts the circles at C and D. Draw any other st. line EF through A cutting the circles at E and F.

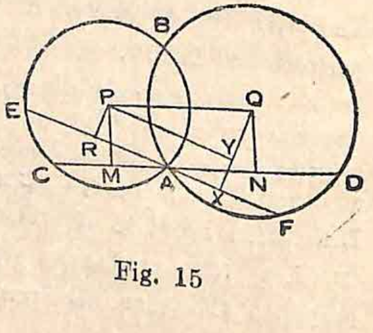


Fig. 15

To prove that $CD>EF$.

Construction : Draw PM and QN perp. to CD and draw PR and QX perp. to EF. Draw $PY \perp QX$.

Proof : $\because$ $PM \perp AC$, $\therefore$ $AM = CM$ $\therefore$ $AM = \frac{1}{2}AC$.
$\because$ $QN \perp AD$, $\therefore$ $AN = \frac{1}{2}AD$, $\therefore$ $MN = \frac{1}{2}CD$. Similarly $RX = \frac{1}{2}EF$.
Now, $PQ = MN = \frac{1}{2}CD$ and $PY = RX = \frac{1}{2}EF$.
$\because$ $\angle PYQ$ is a rt. angle, $\therefore$ $PQ > PY$. $\therefore$ $\frac{1}{2}CD > \frac{1}{2}EF$,
$\therefore$ $CD > EF$.

Ex. 4. To find a point within a triangle such that the sum of its distances from the vertices is a minimum.

Let ABC be a triangle. On AB and AC draw two segments of circles each containing an angle of 120°. Let the segments intersect at O. Then O is the required point.

Proof : Join AO, BO, CO. Through O draw PQ perp. to AO, cutting AB at P and AC at Q. $\because$ $AO \perp PQ$, $\therefore$ AO is the least distance from A to PQ. Again, $\angle BOP = \angle COQ = 30°$.
$\because$ BO and CO make equal angles with PQ at O,
$\therefore$ the sum of the distances $(BO + CO)$ of the pt. O on PQ from B and C is the least. $\therefore$ the sum of the distances of the pt. O from the vertices A, B, C is a minimum.

Some Problems

Ex. 1. Draw an equilateral triangle equal in area to a given triangle. [C. U. '39 Sup. ; '50 (High)]

Let ABC be a triangle. To draw an equilateral triangle equal in area to $\triangle ABC$.

Construction : On BC draw an equilateral triangle DBC. Draw $AE \parallel CB$ and let AE cut BD at E. From BD cut off BF equal to the mean proportional between BE and BD. Draw FH parallel to DC, cutting BC in H.

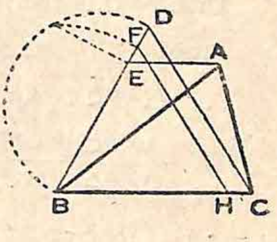


Fig. 16

Then BFH is the required triangle.

Proof : $\because$ FH $\parallel$ DC, $\therefore$ $\angle BFH = \angle D$, $\angle BHF = \angle DCB$,
$\therefore$ $\triangle BFH$ and $\triangle DBC$ are equiangular,

$\therefore$ $\triangle BFH$ is equilateral.

Now, $\dfrac{\triangle BFH}{\triangle BDC} = \dfrac{BF^2}{BD^2} = \dfrac{BE.BD}{BD^2}$ ($\because$ BF is the mean proportional between BE and BD)
$= \dfrac{BE}{BD}$.

Again, joining EC we have, $\dfrac{\triangle BEC}{\triangle BDC} = \dfrac{BE}{BD}$. $\therefore$ $\dfrac{\triangle BFH}{\triangle BDC} = \dfrac{\triangle BEC}{\triangle BDC}$.

$\therefore$ $\triangle BFH = \triangle BCE$; but $\triangle BCE = \triangle ABC$ (being on the same base and between the same parallels),

$\therefore$ $\triangle BFH = \triangle ABC$.

Ex. 2. Construct an equilateral triangle equal in area to a given square.

Let ABCD, be the given square. Produce AB to E so that BE = AB. On AE draw an equilateral triangle AEF and let AF cut DC in H. From AF cut off AP equal to the mean proportional between AH and AF. Draw PQ parallel to FE, cutting AE in Q.

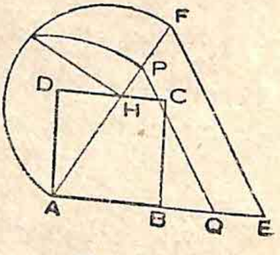


Fig. 17

Then APQ is the required triangle.

Proof : $\because$ PQ $\parallel$ FE, $\therefore$ $\triangle APQ$ and $\triangle AFE$ are equiangular,

$\therefore$ $\triangle APQ$ is equilateral.

$\therefore$ $\dfrac{\triangle APQ}{\triangle AFE} = \dfrac{AP^2}{AF^2} = \dfrac{AH.AF}{AF^2}$ ($\because$ AP is the mean proportional) $= \dfrac{AH}{AF}$. Joining HE we have $\dfrac{\triangle AHE}{\triangle AFE} = \dfrac{AH}{AF}$;

$\therefore$ $\dfrac{\triangle APQ}{\triangle AFE} = \dfrac{\triangle AHE}{\triangle AFE}$, $\therefore$ $\triangle APQ = \triangle AHE$.

Again, $\triangle AHE$ and square ABCD are between the same parallels and the base of the triangle is twice the base of the square. $\therefore$ the equilateral $\triangle APQ = \triangle AHE =$ square ABCD.

Ex. 3. To construct a triangle similar to a given triangle and equal in area to another given triangle.

To draw a triangle equal to the given $\triangle ABC$ and similar to the given $\triangle DEF$.

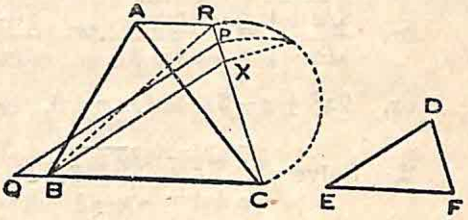


Fig. 18

Construction: Draw $\angle CBX$ and $\angle BCX$ equal respectively to $\angle E$ and $\angle F$, and let BX and CX cut at X. Draw $AR \parallel BC$, cutting CX at R. From CR cut off CP equal to the mean proportional between CX and CR. Draw $PQ \parallel XB$, PQ meeting CB in Q. Then PCQ is the required triangle.

Proof: Join BR. $\because$ $BX \parallel PQ$, $\therefore$ $\triangle BCX$ and $\triangle PCQ$ are equiangular and hence similar.

$$\therefore \quad \frac{\triangle PCQ}{\triangle BCX} = \frac{CP^2}{CX^2} = \frac{CX.CR}{CX^2} = \frac{CR}{CX}.$$

Again, $\dfrac{\triangle BCR}{\triangle BCX} = \dfrac{CR}{CX}$ ($\because$ their heights are equal).

$$\therefore \quad \frac{\triangle PCQ}{\triangle BCX} = \frac{\triangle BCR}{\triangle BCX}. \therefore \triangle PCQ = \triangle BCR = \triangle ABC \ (\because AR \parallel BC).$$

Again, $\because$ $BX \parallel PQ$,

$\therefore$ $\angle PQC = \angle XBC = \angle E$ and $\angle PCQ = \angle F$.

$\therefore$ $\triangle PCQ$ and $\triangle DEF$ are similar.

$\therefore$ $\triangle PCQ$ is equal to $\triangle ABC$ and similar to $\triangle DEF$.

Miscellaneous Exercise

[Algebra]

1. Solve $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\dfrac{\sqrt{2x+3}+\sqrt{2x-3}}{\sqrt{2x+3}-\sqrt{2x-3}}.$

$$\frac{2\sqrt{x+1}}{2\sqrt{x-1}}=\frac{2\sqrt{2x+3}}{2\sqrt{2x-3}} \text{ [by comp. \& div.]}$$

or, $\dfrac{\sqrt{x+1}}{\sqrt{x-1}}=\dfrac{\sqrt{2x+3}}{\sqrt{2x-3}}$, or, $\dfrac{x+1}{x-1}=\dfrac{2x+3}{2x-3}$ [squaring]

or, $2x^2+x-3=2x^2-x-3$, or, $2x=0$, $\therefore$ $x=0$.

2. Solve $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\dfrac{1}{2}.$ [B. U.] [Ans. $x=\frac{5}{4}$]

3. Solve $\dfrac{\sqrt[3]{2x+3}+\sqrt[3]{x-11}}{\sqrt[3]{2x+3}-\sqrt[3]{x-11}}=2.$ [Ans. $x=12$]

4. Solve $y+z=\dfrac{1}{x}$, $z+x=\dfrac{1}{y}$, $x+y=\dfrac{1}{z}.$ [G. U. '49]

[**Hints** : From the equations we have $xy+xz=1 \cdots (1)$, $yz+xy=1 \cdots (2)$ and $xz+yz=1 \cdots (3)$. Adding them we have $2(xy+yz+zx)=3$, or, $xy+yz+zx=\frac{3}{2} \cdots (4)$

From $(4)-(1)$ we have $yz=\frac{1}{2}$ ... ... (5)

„ $(4)-(2)$ „ „ $zx=\frac{1}{2}$ ... ... (6)

„ $(4)-(3)$ „ „ $xy=\frac{1}{2}$ ... ... (7)......

Ans. $x=y=z=\pm\dfrac{1}{\sqrt{2}}$]

5. Solve : $x(2x+3y+4z)=58$ ... ... (1)

$y(2x+3y+4z)=87$ ... ... (2)

$z(2x+3y+4z)=116$ ... ... (3)

Multiplying (1), (2) and (3) by 2, 3 and 4 respectively (*i.e.*, by the coefficients of x, y, z of the expression within brackets) and adding the products, we have

$$(2x+3y+4z)^2=116+261+464=841=(29)^2,$$

$$\therefore \quad 2x+3y+4z=\pm 29 \cdots\cdots(4)$$

Now, dividing (1), (2) and (3) by (4) we have

$$x=\pm 2,\ y=\pm 3,\ z=\pm 4.$$

6. Solve $x^y=y^x \cdots\cdots(1)$, $y^a=x^b \cdots\cdots(2)$. [C. U. '58]

From (1), $x^{\frac{y}{x}}=y^{\frac{x}{x}}=y$ and from (2), $y=x^{\frac{b}{a}}$

$$\therefore \quad x^{\frac{y}{x}}=x^{\frac{b}{a}}, \quad \therefore \quad \frac{y}{x}=\frac{b}{a}, \text{ or, } y=\frac{bx}{a}.$$

Putting $\frac{bx}{a}$ for y in (2) we have

$$\left(\frac{bx}{a}\right)^a=x^b, \text{ or, } \left(\frac{b}{a}\right)^a \times x^a=x^b, \text{ or, } \left(\frac{b}{a}\right)^a=\frac{x^b}{x^a}=x^{b-a}$$

$$\therefore \quad x=\left(\frac{b}{a}\right)^{\frac{a}{b-a}}. \ \therefore \ y=\frac{bx}{a}=\frac{b}{a}\times\left(\frac{b}{a}\right)^{\frac{a}{b-a}}=\left(\frac{b}{a}\right)^{1+\frac{a}{b-a}}=\left(\frac{b}{a}\right)^{\frac{b}{b-a}}$$

$$\therefore \quad x=\left(\frac{b}{a}\right)^{\frac{a}{b-a}},\ y=\left(\frac{b}{a}\right)^{\frac{b}{b-a}}.$$

7. Solve $2^x+3^y=17$ and $2^{x+2}-3^{y+1}=5$. [C. U. '57]
[Ans. $x=3,\ y=2$]

8. Solve $2^{x-1}+3^{y-1}=4$ and $3.2^{x-2}+2.3^{y-3}=2\frac{1}{6}$.
[Ans. $x=1,\ y=2$] [C. U. '58 (C)]

9. Show that the numerical sum of the coefficients of the terms in the expansion of $(a+b)^8$ is 256.

We know that any power of 1 is 1. $\therefore$ if we put $a=1$, $b=1$ in the expansion of $(a+b)^8$, each term in the expansion will evidently be equal to its coefficient.

$$\therefore \quad \text{the sum of the coefficients}=(1+1)^8=2^8=256.$$

10. Find the value of $(2+3\sqrt{-5})^{\frac{1}{2}}+(2-3\sqrt{-5})^{\frac{1}{2}}$

[Ans. $\pm 3\sqrt{2}$] [Pat. '55]

11, If $ma^3-3na^2+3pa-q$ be a perfect cube, show that $n^2=mp$ and $n^3=m^2q$.

Supqose, $ma^3-3na^2+3pa-q=(xa-y)^3$

$=x^3a^3-3x^2ya^2+3xy^2a-y^3$.

$\therefore$ here $m=x^3$, $n=x^2y$, $p=xy^2$, $q=y^3$

$\therefore$ $n^2=x^4y^2=x^3.xy^2=mp$ and $n^3=x^6y^3=m^2q$.

[Trigonometry]

Prove that :—

1. $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta.$

$$\text{L. H. S.}=\sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}+\sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}+\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$=\frac{1+\sin\theta}{\cos\theta}+\frac{1-\sin\theta}{\cos\theta}=\frac{1+\sin\theta+1-\sin\theta}{\cos\theta}=\frac{2}{\cos\theta}=2\sec\theta.$$

2. $\cos\theta+\dfrac{\tan\theta\cos\theta}{\sin\theta}+\dfrac{\sin^2\theta}{\cos\theta}=1+\sec\theta.$ [B. U.]

3. $\dfrac{\sin A+\sec A}{\cos A+\operatorname{cosec} A}=\tan A.$ [B. H. U. '45]

4. $\tan^2 A-\tan^2 B=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}.$ [C. U. '36]

$$\text{L. H. S.}=\frac{\sin^2 A}{\cos^2 A}-\frac{\sin^2 B}{\cos^2 B}=\frac{\sin^2 A\cos^2 B-\sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$=\frac{\sin^2 A(1-\sin^2 B)-\sin^2 B(1-\sin^2 A)}{\cos^2 A\cos^2 B}$$

$$=\frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}.$$

5. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2$ [W. B. S. F. '56]

6. $(\operatorname{cosec}\theta-\cot\theta)^2=\dfrac{1-\cos\theta}{1+\cos\theta}.$ [B. H. U. '40]

7. $\dfrac{\cot\theta}{\operatorname{cosec}\theta-1}-\dfrac{\cos\theta}{1+\sin\theta}=2\tan\theta.$

8. $1-\dfrac{\sin^2\alpha}{1+\cot\alpha}-\dfrac{\cos^2\alpha}{1+\tan\alpha}=\sin\alpha\cos\alpha.$ [B. H. U. '44]

9. $\sec^2\phi\tan\phi+2\sec\phi\operatorname{cosec}\phi+\operatorname{cosec}^2\phi\cot\phi$
$=\sec^3\phi\operatorname{cosec}^3\phi.$ [B. H. U. '42]

$$\text{L. H. S.}=\frac{1}{\cos^2\phi}\cdot\frac{\sin\phi}{\cos\phi}+\frac{2}{\cos\phi\sin\phi}+\frac{1}{\sin^2\phi}\cdot\frac{\cos\phi}{\sin\phi}$$

$$=\frac{\sin^4\phi+2\sin^2\phi\cos^2\phi+\cos^4\phi}{\cos^3\phi\sin^3\phi}=\frac{(\sin^2\phi+\cos^2\phi)^2}{\cos^3\phi\sin^3\phi}$$

$$=\frac{1}{\cos^3\phi\sin^3\phi}=\sec^3\phi\operatorname{cosec}^3\phi.$$

10. $\dfrac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta}=1-2\sin^2\theta.$

11. $\dfrac{\sin A}{\cos A+\sin B}+\dfrac{\sin B}{\cos B-\sin B}$

$$=\frac{\sin A}{\cos A-\sin B}+\frac{\sin B}{\cos B+\sin A}.$$

12. $\dfrac{1+\sec^2A\cot^2B}{1+\sec^2C\cot^2B}==\dfrac{1+(\tan A\cos B)^2}{1+(\tan C\cos B)^2}$

[**Hints :** $\text{L. H. S.}=\dfrac{1+(1+\tan^2A)\cot^2B}{1+(1+\tan^2C)\cot^2B}$

$$=\frac{1+\cot^2B+\tan^2A\cot^2B}{1+\cot^2B+\tan^2C\cot^2B}\ldots$$

Now change tan and cot into $\dfrac{\sin}{\cos}$ and $\dfrac{\cos}{\sin}$.]

13. If $x=\dfrac{1+\sin\theta}{\cos\theta}$, show that $\dfrac{1}{x}=\dfrac{1-\sin\theta}{\cos\theta}.$

14. If $4 \sec^2 \alpha - 7 \tan^2 \alpha = 3$, prove that $\sin \alpha = \pm \frac{1}{2}$. [B. H. U. '40]

Here, $4 \sec^2 \alpha - 7 \tan^2 \alpha = 3$,

or, $4(1+\tan^2 \alpha) - 7 \tan^2 \alpha = 3$, *or*, $4 - 3 \tan^2 \alpha = 3$,

or, $3 \tan^2 \alpha = 1$, $\therefore \tan^2 \alpha = \frac{1}{3}$,

or, $\frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1}{3}$, *or*, $\frac{\sin^2 \alpha}{1 - \sin^2 \alpha} = \frac{1}{3}$, *or*, $4 \sin^2 \alpha = 1$,

or, $\sin^2 \alpha = \frac{1}{4}$, $\therefore \sin \alpha = \pm \frac{1}{2}$.

15. Given $\sec \theta + \tan \theta = u$, express $\tan \theta$ in terms of u.

$\left[\text{Ans. } \tan \theta = \frac{u^2 - 1}{2u} \right]$ [P. U. '42]

16. If $\tan A + \sin A = a$, and $\tan A - \sin A = b$, then $a^2 - b^2 = 4 \sqrt{ab}$. [B. H. U. '39]

17. If $\tan A = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$,

show that $\sqrt{2} \sin A = \sin \theta - \cos \theta$.

18. If $x \sin \alpha - y \cos \alpha = z$, then $x \cos \alpha + y \sin \alpha$

$= \pm \sqrt{x^2 + y^2 - z^2}$.

[**Hints** : Squaring both sides of $x \sin \alpha - y \cos \alpha = z$, add $-x^2 - y^2$ to both sides.]

19. If $\tan^2 A = 1 + 2 \tan^2 B$,

prove that $2 \sin^2 A = 1 + \sin^2 B$.

20. If $\cos \theta + \sec \theta = 2$, find the value of $\cos^n \theta + \sec^n \theta$.

Here, $\cos \theta + \sec \theta = 2$,

or, $\cos \theta + \frac{1}{\cos \theta} = 2$, *or*, $\cos^2 \theta + 1 = 2 \cos \theta$,

or, $\cos^2 \theta - 2 \cos \theta + 1 = 0$, *or*, $(\cos \theta - 1)^2 = 0$, $\therefore \cos \theta = 1$.

$\therefore \sec \theta = \frac{1}{\cos \theta} = 1$. Now. $\cos^n \theta + \sec^n \theta = (1)^n + (1)^n = 2$.

21. If $\operatorname{cosec} \alpha + \sin \alpha = 2$, show that $\sin^7 \alpha + \operatorname{cosec}^7 \alpha = 2$.
[W. B. S. F. '60]

22. If $y \tan A = x$, find the value of $\dfrac{x \sin A - y \cos A}{x \sin A + y \cos A}$.

$\because \quad y \tan A = x, \quad \therefore \quad xy \tan A = x^2$ (multiplying by x)

$$\therefore \quad x \tan A = \frac{x^2}{y}.$$

Now, $\dfrac{x \sin A - y \cos A}{x \sin A + y \cos A} = \dfrac{x \tan A - y}{x \tan A + y}$

[dividing the numerator and denominator by cos A]

$$= \frac{\dfrac{x^2}{y} - y}{\dfrac{x^2}{y} + y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

23. If $\tan \theta + \cot \theta = 2$, find $\sin \theta$, θ being a positive acute angle. [W. B. S. F. '55] $\left[\text{Ans. } \dfrac{1}{\sqrt{2}}\right]$

21. If $\tan \theta = 2 \sin \theta$, where θ is a positive acute angle, find $\cot \theta$. [W. B. S. F. '57] $\left[\text{Ans. } \dfrac{1}{\sqrt{3}}\right]$

22. Given $\tan^2 \theta = 1 - e^2$, show that $\sec \theta + \tan^3 \theta \operatorname{cosec} \theta = (2 - e^2)^{\frac{3}{2}}$. [W. B. S. F. '59]

$$\because \quad \sec^2 \theta = 1 + \tan^2 \theta = 1 + 1 - e^2 = 2 - e^2,$$

$$\therefore \quad \sec \theta = (2 - e^2)^{\frac{1}{2}}$$

$$\therefore \quad \sec \theta + \tan^3 \theta \operatorname{cosec} \theta = \sec \theta + \tan^2 \theta . \frac{\sin \theta}{\cos \theta} . \frac{1}{\sin \theta}$$

$$= \sec \theta + \tan^2 \theta . \frac{1}{\cos \theta} = \sec \theta + \tan^2 \theta \sec \theta = \sec \theta \, (1 + \tan^2 \theta)$$

$$= \sec^3 \theta = \left\{(2 - e^2)^{\frac{1}{2}}\right\}^3 = (2 - e^2)^{\frac{3}{2}}.$$

26. Find α and β in degrees, (α and β being positive acute angles) if $\sin(2\alpha-\beta)=1$, and $\cos(\alpha+\beta)=\frac{1}{2}$.

[W. B. S. F. '59] [Ans. $\alpha=50°, \beta=10°$]

27. If $\cos^2\theta-\sin^2\theta=\tan^2\alpha$,

prove that $\cos^2\alpha-\sin^2\alpha=\tan^2\theta$. [W. B. S. F. '60]

$$\because \cos^2\theta-\sin^2\theta=\tan^2\alpha, \therefore \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\frac{\tan^2\alpha}{1}=\frac{\sin^2\alpha}{\cos^2\alpha}$$

$[\because \cos^2\theta+\sin^2\theta=1]$

$\therefore$ By comp. & div. we have $\dfrac{2\cos^2\theta}{-2\sin^2\theta}=\dfrac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha-\cos^2\alpha}$

or, $\dfrac{\cos^2\theta}{-\sin^2\theta}=\dfrac{1}{\sin^2\alpha-\cos^2\alpha}$, or, $\dfrac{\cos^2\theta}{\sin^2\theta}=\dfrac{1}{\cos^2\alpha-\sin^2\alpha}$

or, $\cos^2\alpha-\sin^2\alpha=\dfrac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$.

28. If $6\sin^2\theta-11\sin\theta+4=0$, where θ is a positive acute angle, find θ in degrees. [Ans. 30°]

29. If $\sin^2\theta+\sin^4\theta=1$, prove that $\tan^4\theta-\tan^2=1$.

[W. B. S. F. '59 (C)]

30. If $5\tan A=4$, find the value of $\dfrac{5\sin A-3\cos A}{\sin A+2\cos A}$.

[W. B. S. F. '60] [Ans. $\frac{5}{14}$]

31. Solve for θ, when θ is an acute angle :

$$\frac{1-\tan\theta}{1+\tan\theta}=\frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

By comp. & div. we have $\dfrac{2}{-2\tan\theta}=\dfrac{2\sqrt{3}}{-2}$,

or, $\dfrac{1}{\tan\theta}=\sqrt{3}$, or, $\tan\theta=\dfrac{1}{\sqrt{3}}=\tan 30°$, $\therefore \theta=30°$.

32. Solve $\tan\alpha+\sqrt{3}\cot\alpha=\sqrt{3}+1$, where α is an acute angle.

[Ans. $\alpha=60°$ or $45°$]

33. The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 60 to π. Find the angles in degrees.

[W. B. S. F. '60; Ans. $30^\circ, 60^\circ, 90^\circ$]

34. Assuming the earth to be a sphere of radius 4000 miles, determine approximately the difference in latitude (to be expressed in the sexagesimal system) of two places, one of which lies 200 miles due north of the other.

$\left(\frac{1}{\pi} = {\cdot}31831.\right)$ [W. B. S. F. '56]

[**Hints :** arc $= r\theta$; here arc $= 200$ miles, $r = 4000$ miles. To find θ from the formula arc $= r\theta$, where θ is in radian measure.

$$\therefore\ 4000 \times \theta = 200, \quad \therefore\ \theta = \frac{200}{4000} \text{ radian} = \frac{1}{20} \times \frac{180^\circ}{\pi}$$

$$= 9^\circ \times \frac{1}{\pi} = 9^\circ \times {\cdot}31831 = 2^\circ 51' 53{\cdot}244'']$$

35. An arc of 50^g in one circle equals one of 60° in another ; find the radian measure of an angle subtended at the centre of the first circle by an arc equal to the radius of the second.

[W. B. S. F. '56]

$$50^g = \frac{\pi^c}{4} \text{ and } 60^\circ = \frac{\pi^c}{3}.$$

Let R and r be respectively the radii of the two circles.

Then, in the first circle, arc $= \frac{\pi}{4}$R, and in the second, arc $= \frac{\pi}{3}r$. $\therefore\ \frac{\pi}{3}r = \frac{\pi}{4}\text{R}$, $\therefore\ \frac{r}{\text{R}} = \frac{3}{4}$.

Now, the radian measure of the angle subtended at the centre of the first circle by an arc equal to r

$$= \frac{\text{arc}}{\text{radius}} \text{ radian} = \frac{r}{\text{R}} \text{ radian} = \frac{3}{4} \text{ radian}.$$

36. The height of a house subtends a right angle at an opposite window, the top of the house being 60° above a horizontal line through the window ; find the height of the house, taking the breadth of the street to be 30 ft.

(W. B. S. F. '59) [Ans. $40\sqrt{3}$ ft.]

37. Two pillars are respectively 180 and 60 feet high. If the angle of elevation of the top of the first from the foot of the second be 60°, what will be the angle of elevation of the top of the second from the foot of the first ? (W. B. S. F. '60) [Ans. 30°]

38. Show that :—

(*a*) $1+\tan A+\tan^2 A+\tan^3 A=\dfrac{\sin A+\cos A}{\cos^3 A}$.

(*b*) $1+\cot A+\cot^2 A+\cot^3 A=\dfrac{\sin A+\cos A}{\sin^3 A}$.

(*c*) $\sec^6\theta-\tan^6\theta=1+3\sec^2\theta\tan^2\theta$.

(*d*) $\operatorname{cosec}^6\theta-\cot^6\theta=1+3\operatorname{cosec}^2\theta\cot^2\theta$.

39. If $\tan\theta+\text{cet}\,\theta=2$, show that $\tan^7\theta+\cot^7\theta=2$.

40. If $p\sin\theta+q\cos\theta=r$, and $p\,\text{ces}\,\theta-q\sin\theta=s$,

then $p^2+q^2=r^2+s^2$ and $\tan\theta=\dfrac{r-q\cos\theta}{s+q\sin\theta}$.

[Geometry]

1. PQRS is a parallelogram ; A and B are two points outside it and are such that AB is parallel to PQ ; AP and BQ when produced meet at C and AS and BR when produced meet at D ; show that CD is parallel to PS. [Applied Math. (Tech.), 1960]

2. O is the in-centre of the $\triangle$ABC and AO produced meets BC at D. Show that AO : OD $=$ (AB $+$ AC) : BC. [Tech. (1960)]

3. In the quadrilateral ABCD, BC $=$ 2AD and O is a point on the diagonal AC such that CO $=$ 2AO. If BO $=$ 2DO, prove that ABCD is a trapezium and O lies on the diagonal BD. (G. U. '58]

4. The adjacent sides of a rectangular land are 20 ft. and 8 ft. respectively ; calculate geometrically the side of the square whose area will be equal to that of the rectangular plot.
[Tech. 1960]

5. Two circles intersect at A and B. Prove that of the similar triangles having a straight line drawn through A and intercepted by the circumferences as their base and B as their vertex, the area of the triangle whose base is perpendicular to AB is the maximum.

[**Hints** : (Draw a fig.). Let PBQ, XBY be two such similar triangles. Let $PQ \perp AB$ and draw $BR \perp XY$; then $AB > BR$ and AB BR are respectively the altitudes of $\triangle^s$ PBQ, XBY.

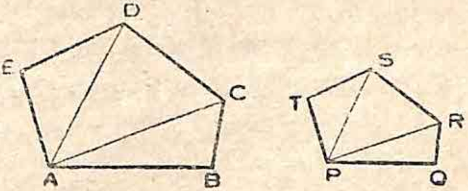
$\because$ The triangles are similar, $\therefore$ the ratio of their areas $= AB^2 : BR^2$. $\therefore$ $\triangle PBQ > \triangle XBY \cdots$]

6. DEF is the pedal triangle of the $\triangle ABC$; prove that $\triangle ABC : \triangle CDE = BC^2 : CE^2$.

7. Bisect a triangle by a st. line drawn parallel to a straight line given in position.

8. Areas of similar polygons are proportional to the squares of the corresponding sides.

Let ABCDE and PQRST be two similar polygons in which angles A, B, C, D and E are respectively equal to angles P, Q, R, S and T and AB, PQ are corresponding sides.



To prove that $\dfrac{\text{polygon ABCDE}}{\text{polygon PQRST}} = \dfrac{AB^2}{PQ^2}$.

Join AC, AD and PR, PS.

Proof. $\because$ the polygons are similar,

$\therefore$ $\angle B = \angle Q$, and $AB : PQ = BC : QR$,

$\therefore$ $\triangle ABC$ and $\triangle PQR$ are similar.

$\therefore$ $\angle ACB = \angle PRQ$, and $\frac{AC}{PR} = \frac{BC}{QR} = \frac{DC}{RS}$ ($\because$ Polygons are similar)

$\because$ $\angle BCD = \angle QRS$ and $\angle ACB = \angle PRQ$, $\therefore \angle ACD = \angle PRS$

Now, in $\triangle^s$ ACD and PRS, $\angle ACD = \angle PRS$ and the sides about them are proportional, $\therefore$ the triangles are similar. Similarly $\triangle AED$ and $\triangle PST$ are similar.

$$\therefore\ \frac{\triangle ABC}{\triangle PQR} = \frac{AC^2}{PR^2} = \frac{\triangle ADC}{\triangle PRS}; \text{ and } \frac{\triangle ADC}{\triangle PRS} = \frac{AD^2}{PS^2} = \frac{\triangle ADE}{\triangle PST}.$$

$$\therefore\ \frac{\triangle ABC}{\triangle PQR} = \frac{\triangle ADC}{\triangle PRS} = \frac{\triangle ADE}{\triangle PST} = \frac{\triangle ABC + \triangle ADC + \triangle ADE}{\triangle PQR + \triangle PRS + \triangle PST}$$

$$= \frac{\text{polygon } ABCDE}{\text{polygon } PQRST},$$

$$\therefore\ \frac{\text{polygon } ABCDE}{\text{polygon } PQRST} = \frac{\triangle ABC}{\triangle PQR} = \frac{AB^2}{PQ^2}.$$

QUESTIONS

SET IN HIGHER SECONDARY (W. BENGAL) EXAMINATIONS WITH ANSWERS

[Pretaining to *Syllabus* for class IX only]

HIGHER SECONDARY EXAMINATION—1960

FIRST PAPER

GROUP A—**Algebra**

1. (*a*) If $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$, $y=\frac{\sqrt{3}-1}{\sqrt{3}+1}$, find the value of

$\frac{x^2+xy+y^2}{x^2-xy+y^2}$. [Ans. $\frac{15}{13}$]

(*b*) Simplify : $\left[\sqrt[3]{4}\times\frac{1}{\sqrt[9]{8}}\times\sqrt[12]{16^{-1}}\right]^{\frac{1}{2}}$ [Ans. 1]

(*c*) Find the square root of $28-6\sqrt{3}$. [Ans. $\pm(3\sqrt{3}-1)$]

2, (*b*) Solve : $\left.\begin{array}{l}3x+4y=5xy\\2y+3z=2yz\\5z+2x=6zx\end{array}\right\}$ [Ans. $x=y=z=0$; or, $x=2$, $y=3$, $z=2$]

GROUP B—**Trigonometry**

9. (*a*) Prove that the radian is a constant angle. Find its value in degrees, minutes, etc. [$\pi=\frac{22}{7}$] [Ans. 57°17′44·8″]

(*b*) The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 60 is to π. Find the angles in degrees, [Ans. 30°, 60°, 90°]

13. Two vertical pillars the height of one of which is double that of the other are at a distance of 150 ft. from each other. At a point between the pillars and on the line joining their feet the angular elevations of the tops of the taller and the shorter pillar are found to be 60° and 30° respectively. Find the heights of the pillars and the position of the point.

[Ans. Heights $=60\sqrt{3}$ ft., $30\sqrt{3}$ ft., distance $=60$ ft. from the taller tower.]

SECOND PAPER

1. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*).

(*a*) Prove that in any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle, diminished by twice the rectangle contained by one of these sides and the projection on it of the other side.

(*b*) Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

(*c*) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) A straight line AB is divided in a given ratio internally at C and externally at D. If P be a point where CD subtends a right angle, prove that PC bisects the angle APB.

HIGHER SECONDARY EXAMINATION

Compartmental—1960

FIRST PAPER

GROUP A—Algebra

1. (*a*) Given $\sqrt{2}=1{\cdot}414$, find to three decimal places, the value of $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$.

[Ans. 14·070]

(*b*) Simplify : $\left[81^{-\frac{3}{4}}\times\frac{16^{\frac{1}{4}}}{6^{-2}}\times\left(\frac{1}{27}\right)^{-\frac{4}{3}}\right]^{\frac{1}{3}}$

[Ans. 6]

(*c*) Find the square root of $17+12\sqrt{2}$. [Ans. $\pm(3+2\sqrt{2})$]

GROUP B—Trigonometry

8. (*a*) The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians; express these angles in degrees. [Ans. $81^\circ, 9^\circ$]

(*b*) If s is the length of the arc of a circle whose radius is r and θ is the radian measure of the angle at the centre, standing on the arc, prove that $\theta = \frac{s}{r}$.

9. (*b*) If $\sin A = \frac{3}{5}$ end $\cos B = \frac{12}{13}$ where A and B are acute angles, find the value of $\frac{\tan A - \tan B}{1 + \tan A \tan B}$. $\left[\text{Ans. } \frac{16}{63}\right]$

12. The upper part of a straight tree broken over by the wind but not completely separated, makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet. What was the height of the tree? [Ans. $50\sqrt{3}$ ft.]

SECOND PAPER

1. Answer *either* (*a*) and (*b*), *or* (*c*) and (*d*).

(*a*) If two triangles are equiangular, prove that their corresponding sides are proportional.

(*b*) Prove that the line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at the point.

(*c*) Prove that in a triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

(*d*) Show that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

HIGHER SECONDARY EXAMINATION—1961

FIRST PAPER

Group A—Algebra

1. (*a*) Simplify : $\frac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

[Ans. $\sqrt{3}$ or, 1·732]

(*b*) Simplify : $\sqrt[ln]{\frac{x^l}{x^n}}\times\sqrt[mn]{\frac{x^n}{x^m}}\times\sqrt[lm]{\frac{x^m}{x^l}}$. [Ans. 1]

(*c*) Find the sq. root of $33-4\sqrt{35}$. [Ans. $\pm(2\sqrt{7}-\sqrt{5})$]

Group B—Trigonometry

8. (*a*) The radius of a circle is 10 cm., find the angle in degrees and minutes, subtended at its centre by an arc 6 cm. in length. [$\pi=\frac{22}{7}$] [Ans. 34°21·8′]

(*b*) The angles of a triangle are in A.P. If the number of degrees in the greatest angle is the same as the number of grades in the least, find the angles in degrees. [Ans. $63^\circ\frac{3}{19}$, 60°, $55^\circ\frac{16}{19}$]

12. On a straight coast there are three objects A, B and C such that $AB=BC=4$ miles. A steamer approches B in a line perpendicular to the coast and at a certain point AC is found to subtend an angle 60° ; after sailing in the same direction for ten minutes, AC is found to subtend an angle of 120° ; find the rate at which the steamer is going. [Ans. $16\sqrt{3}$ miles per hr.].

SECOND PAPER

1. (*a*) If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, prove that the triangles are similar.

(*b*) If two triangles are similar, prove that their areas are proportional to the squares on their corresponding medians.

(*c*) Prove that the ratio of the areas of similar triangles is equal to the ratio of the squares on their corresponding sides.

H. S. Exam. (Compartmental)—1961

FIRST PAPER

Group A—Algebra

1. (*a*) Simplify :—

$$\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$$ [Ans. 0]

(*b*) Find the square root of $37-20\sqrt{3}$. [Ans. $\pm(5-2\sqrt{3})$]

Group B—Trigonometry

8. (*a*) Define a radian. Taking $\pi=3·1416$, show that a radian contains 206265 seconds apprpximately.

(*b*) One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{2}{3}x$ degrees, whilst the third is $\frac{x\pi}{75}$ radians ; express them all in degrees. [Ans. 24°, 60°, 96°]

7. Find the value of

$$\frac{2\tan^2 30^\circ}{1-\tan^2 30^\circ}+(\sec^2 45^\circ-\cot^2 45^\circ)-(\cos^2 60^\circ+\sin^2 120^\circ).$$

[Ans. 1]

12. Two vertical poles are 120 feet apart and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary. Find their heights.

[Ans. $30\sqrt{2}$ ft., $60\sqrt{2}$ ft.]

SECOND PAPER

1. Answer *either* (*a*) and (*b*), or (*c*) and (*d*) :—

(*a*) Prove that the bisector of the exterior angle of a triangle divides the opposite side externally in the ratio of the other two sides.

(*b*) In a quadrilateral, if the bisectors of one pair of opposite angles meet on one diagonal, prove that the bisectors of the other pair of opposite angles will meet on the other diagonal.

(*c*) If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, prove that the triangles on each side of the perpendicular are similar to one another Hence deduce that the perpendicular is a mean proportional between the segments of the hypotenuse.

(*d*) In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, show that the segments of the hypotenuse have the same ratio as the squares on the sides containing the right angle.

HIGHER SECONDARY EXAMINATION, 1962

FIRST PAPER—Group A—Algebra

1. (*a*) Given $x=\frac{\sqrt{3}}{2}$, find the value of

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \qquad [\text{ Ans. } \frac{1}{\sqrt{3}}\,]$$

(*b*) Find the square root of $17-12\sqrt{2}$. [Ans. $\pm(3-2\sqrt{2})$]

(*c*) Simplify : $(8x^3 \div 27a^{-3})^{\frac{2}{3}} \times (64x^3 \div 27a^{-3})^{-\frac{2}{3}}$. [Ans. $\frac{1}{4}$]

Group B—Trigonometry

8. (*a*) The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. Find the number of degrees and minutes in the third angle. [$\frac{22}{7}$ radians $=2$ right angles]. [Ans. $132^\circ 16'\cdot\dot{3}\dot{6}$]

(*b*) The diameter of a graduated circle is 6 ft. and the graduation on its rim are $15'$ apart ; find the distance (in inches correct to two places of decimals) from one graduation to another next to it. [$\pi=\frac{22}{7}$] [Ans. ·16 in.]

12. The angle of elevation of the top of a tower is observed to be 60° from a point in the horizontal plane through the foot of the tower ; at the point 40 ft. vertically above the first point of observation, the elevation is found to be 45°. Find the height of the tower and its horizontal distance from the points of observation. [Ans. $h=94\cdot6$ ft., $d=54\cdot6$ ft.]

SECOND PAPER

Group A—**Plane Geometry**

1. (*a*) Prove that in an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the suuares on the sides containing the obtuse angle, together with twice the rectangle contained by one of these sides and the projection of the other side on it.

(*b*) Same Q. 1 (*d*) of H. S. 1960 (Compl.).

3. (*a*) Same as Q. 1(*a*) of H. S. 1960 (Compl.).

(*b*) In the trapezium *ABCD*, *AB* is parallel to *DC* and the diagonals intersect at *O*. Show that $OA : OC = OB : OD$.

4. (*a*) Same as Q. 1 (*c*) of H. S. 1960.

(*b*) *AD* is a median of the triangle *ABC*, and the angles *ADB*, *ADC* are bisected by lines which meet *AB*, *AC* at *E* and *F* respectively. Show that *EF* is parallel to *BC*.

H. S. Exam. (Compartmental)—1962

FIRST PAPER

Group A—**Algebra**

1. (*a*) Simplify : $\dfrac{3\sqrt{8}-2\sqrt{12}+\sqrt{20}}{3\sqrt{18}-2\sqrt{27}+\sqrt{45}}$. [Ans. $\frac{2}{3}$]

(*b*) Find the square root of $8+\sqrt{60}$. [Ans. $\pm(\sqrt{5}+\sqrt{3})$]

(*c*) Simplify : $\left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} \times \left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}}$. [Ans. 1]

Group B—**Trigonometry**

8. (*a*) Same as Q. 9(*a*) of H. S. 1960.

(*b*) Find the ratio of the radii of two circles, at the centres of which, two arcs of the same length subtend angles of 60° and 75°. [Ans. 5 : 4]

12. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 50 feet. What was the height of the tree ? [Ans. $50\sqrt{3}$ ft.]

SECOND PAPER

Group A—**Plane Geometry**

1. (*a*) Same as Q. 1(*a*) of H. S. 1960

(*b*) ABC is an isosceles triangle in which $AB=AC$; and BE is drawn perpendicular to AC. Shew that $BC^2=2AC.CE$.

3. (*a*) Same as Q. 1(*c*) of H. S. 1961.

(*b*) A trapezium $ABCD$ has its sides AB, CD parallel and its diagonals intersect at O. If AB is double of CD, find the ratio of the areas of the triangles AOB and COD. [Ans. 4 : 1]

4. (*a*) Same as Q. 1(*c*) of H. S. 1961 (Compl.).

H. S. Examination—1963

FIRST PAPER

Group A—**Algebra**

1. (*a*) Simplify :—

$$\sqrt{\frac{(\sqrt{12}-\sqrt{8})(\sqrt{3}+\sqrt{2})}{5+\sqrt{24}}}.$$ [Ans. $\sqrt{6}-2$]

(*b*) If $a^x=m$, $a^y=n$ and $a^2=(m^y n^x)^z$, prove that $xyz=1$.

(*c*) Find the square root of $\frac{2+\sqrt{3}}{2}$. [Ans. $\pm\frac{1}{2}(\sqrt{3}+1)$]

Group B—**Trigonometry**

8. *b*) If $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$, prove that
$$\cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

11. A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is found to be 60° ; assuming that the boat moves uniformly, how soon will it reach the shore ? [Ans. $1\frac{1}{2}$ min.]

SECOND PAPER

Group A—Plane Geometry

1. (*a*) If two triangles have their sides proportional, when taken in order, prove that they are equiangular.

(*b*) Prove that the areas of two similar triangles are proportional to the squares on their circum-radii.

2. (*a*) If the base of a triangle be divided externally in the ratio of the other two sides, prove that the line joining the vertex to this point of division bisects the vertical angle externally.

(*b*) Prove that the external bisectors of two angles and the internal bisector of the third of a triangle are concurrent.

H, S. Exam.—(Compartmental)—1963

FIRST PAPER

Group A—Algebra

1. (*a*) Simplify :—$\dfrac{\sqrt{2}(\sqrt{3}+1)(2-\sqrt{3})}{(\sqrt{2}-1)(3\sqrt{3}-5)(2+\sqrt{2})}$.

[Ans. $2+\sqrt{3}$]

(*b*) Simplify : $\left\{\sqrt[3]{4}\times\dfrac{1}{\sqrt[6]{8}}\times\sqrt[12]{2^{-1}}\right\}^{\frac{1}{4}}$. $\left[\text{Ans. } 2^{\frac{1}{48}}\right]$

(*c*) Find the square root of $\sqrt{18}-\sqrt{16}$.

[Ans. $\pm\sqrt[4]{2}(\sqrt{2}-1)$]

Group B—Trigonometry

11. At the foot of a mountain, the elevation of its top is found to be 45° ; after proceeding one mile towards the top up a slope of 30° inclination to the horizontal, the elevation of the top is found to be 60°. Find the height of the mountain.

[Ans. $\frac{1}{2}(\sqrt{3}+1)$*mi.* or 1·37 miles (App.)]

SECOND PAPER

Group A—**Plane Geometry**

1. (*a*) Prove that in any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

ABCD is a rectangle, and *P* is any point within it. Prove that $PA^2+PC^2=PB^2+PD^2$.

2. (*a*) If a straight line is drawn parallel to one side of a triangle, prove that the other two sides are divided proportionally.

(*b*) *ABCD* is a trapezium in which *AB* is parallel to *DC*. If the diagonals *AC*, *BD* intersect at *O*, and through *O*, the straight line *POQ* be drawn parallel to *AB* or *DC*, meeting *AD*, *BC* at *P* and *Q* respectively, prove that *PQ* is bisected at *O*.

HIGHER SECONDARY EXAMINATION, 1964

FIRST PAPER

GROUP A—**Algebra**

1. (*a*) Simplify : $\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$. [Ans. 0]

(*b*) If $x^{\frac{1}{a}}=y^{\frac{1}{b}}=z^{\frac{1}{c}}$ and $xyz=1$, prove that $a+b+c=0$.

GROUP B—**Trigonometry**

11. Two chimneys are of equal height. A person standing between them in the line joining their bases, which is horizontal, observes the elevation of the nearer one to be 60°. After walking 80 feet in a horizontal direction perpendicular to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height of the chimneys.

[Ans. $40\sqrt{6}$ ft. or $20\sqrt{6}$ ft.]

SECOND PAPER

GROUP A—**Plane Geometry**

2. Same as Q. 1 (*a*) of H. S. 1960.

ABC is an isosceles triangle and *AY* is drawn to cut the base internally at *Y*. Show that $AY^2 = AB^2 - BY.YC$.

3. Same as Q. 1(*a*) of H. S. 1960 (Compl.).

Prove that the altitudes of two similar triangles are proportional to the corresponding sides.

H. S. Exam. (Compartmental), 1964

FIRST PAPER

GROUP A—**Algebra**

1. (*a*) Simplify : $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)} - \dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$. $\left[\text{Ans. } \dfrac{\sqrt{6}}{3}\right]$

(*b*) Simplify : $\left\{\dfrac{4^{m+\frac{1}{4}} \times \sqrt{2.2^m}}{2.\sqrt{2^{-m}}}\right\}^{\frac{1}{m}}$ [Ans. 8]

(*c*) If $x = 7+4\sqrt{3}$, find the value of $\sqrt{x} - \dfrac{1}{\sqrt{x}}$. [Ans. $2\sqrt{3}$]

GROUP B—**Trigonometry**

8. (*b*) If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$, prove that

$$m^2 - n^2 = 4\sqrt{mn}.$$

11. From an observation balloon 7,200 *ft.* above the sea, the angle of depression of two cruisers are 30° and 45°. Find the distance detween the cruisers if one is east and the other south of the balloon. [Ans. 14400 ft.]

SECOND PAPER

Group A—**Plane Geometry**

1. (*a*) Prove that in any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

$ABCD$ is a rectangle, and P is any point within it. Prove that $PA^2+PC^2=PB^2+PD^2$.

2. (*a*) If a straight line is drawn parallel to one side of a triangle, prove that the other two sides are divided proportionally.

(*b*) $ABCD$ is a trapezium in which AB is parallel to DC. If the diagonals AC, BD intersect at O, and through O, the straight line POQ be drawn parallel to AB or DC, meeting AD, BC at P and Q respectively, prove that PQ is bisected at O.

HIGHER SECONDARY EXAMINATION, 1964

FIRST PAPER

GROUP A—**Algebra**

1. (*a*) Simplify : $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$. [Ans. 0]

(*b*) If $x^{\frac{1}{a}}=y^{\frac{1}{b}}=z^{\frac{1}{c}}$ and $xyz=1$, prove that $a+b+c=0$.

GROUP B—**Trigonometry**

11. Two chimneys are of equal height. A person standing between them in the line joining their bases, which is horizontal, observes the elevation of the nearer one to be 60°. After walking 80 feet in a horizontal direction perpendicular to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height of the chimneys.

[Ans. $40\sqrt{6}$ ft. or $20\sqrt{6}$ ft.]

SECOND PAPER

GROUP A—Plane Geometry

2. Same as Q. 1 (*a*) of H. S. 1960.

ABC is an isosceles triangle and AY is drawn to cut the base internally at Y. Show that $AY^2 = AB^2 - BY.YC$.

3. Same as Q. 1(*a*) of H. S. 1960 (Compl.).

Prove that the altitudes of two similar triangles are proportional to the corresponding sides.

H. S. Exam. (Compartmental), 1964

FIRST PAPER

GROUP A—Algebra

1. (*a*) Simplify : $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)} - \dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$. $\left[\text{Ans. } \dfrac{\sqrt{6}}{3}\right]$

(*b*) Simplify : $\left\{\dfrac{4^{m+\frac{1}{4}} \times \sqrt{2.2^m}}{2.\sqrt{2^{-m}}}\right\}^{\frac{1}{m}}$ [Ans. 8]

(*c*) If $x = 7 + 4\sqrt{3}$, find the value of $\sqrt{x} - \dfrac{1}{\sqrt{x}}$. [Ans. $2\sqrt{3}$]

GROUP B—Trigonometry

8. (*b*) If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$, prove that

$$m^2 - n^2 = 4\sqrt{mn}.$$

11. From an observation balloon 7,200 ft. above the sea, the angle of depression of two cruisers are 30° and 45°. Find the distance between the cruisers if one is east and the other south of the balloon. [Ans. 14400 ft.]

SECOND PAPER, 1964 (Compl.)

GROUP A—Plane Geometry

3. Same as Q. 1 (c) of H. S., 1961.

XY is drawn parallel to BC, the base of the triangle ABC, cutting the sides AB, AC at X, Y respectively. Given that $AX=3$ in., $XB=1{\cdot}8$ in. Calculate the ratio of the area of the triangle AXY to that of the quadrilateral $BCYX$. [Ans. 25 : 39]

4. Same as Q. 1(c) of H. S., 1960.

Hence prove that in any triangle, the internal bisectors of the three angles are concurrent.

H. S. Examination, 1965

FIRST PAPER

GROUP A—Algebra

1. (a) If $x=(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}$, find the value of $x^3+3bx-2a$. [Ans. 0]

(b) Simplify : $\left\{81^{-\frac{3}{4}}\times\frac{16^{\frac{1}{4}}}{6^{-2}}\times\left(\frac{1}{27}\right)^{-\frac{4}{3}}\right\}^{\frac{1}{3}}$. [Ans. 6]

(c) Find the square root of $18+6\sqrt{5}$. [Ans. $\pm(\sqrt{15}+\sqrt{3})$]

SECOND PAPER

GROUP A—Plane Geometry

2. (a) Same as Q. 2(a) of H. S., 1963 (Compl.).

(b) Prove that the straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

4. Same as Q. 1(c) of H. S., 1961 (Compl.).

ABC is a triangle right-angled at A and AC' is drawn perpendicular to the hypotenuse ; also $C'A'$ is drawn parallel to CA. If $AC=15$ cm. and $AB=20$ cm. find, geometrically, the lengths of AC' and $C'A'$. [Ans. $AC'=12$ cm., $A'C'=9{\cdot}6$ cm.]

H. S. Examination, 1966

FIRST PAPER

Group A—Algebra

1. (*c*) Simplify : $\frac{3\sqrt{7}}{\sqrt{5}+\sqrt{2}}-\frac{5\sqrt{5}}{\sqrt{2}+\sqrt{7}}+\frac{2\sqrt{2}}{\sqrt{7}+\sqrt{5}}$. [Ans. 0]

Group B—Trigonometry

12. (*b*) The shadow of a tower standing vertically on a horizontal plane is 50 ft. longer when the altitude of the sun is 30° than when it is 45°. Find the height of the tower.

[Ans. $25(\sqrt{3}+1)$ ft.]

SECOND PAPER

Group A—Plane Geometry

2. First part :—Same as Q. 1(*a*) of H. S., 1962.

The medians of a triangle ABC meet at O. Prove that

$$AB^2+BC^2+CA^2=3(AO^2+BO^2+CO^2),$$

HIGHER SECONDARY EXAMINATION, 1967

FIRST PAPER

Group A—Algebra

1. (*a*) Simplify :—

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}.$$ [Ans. $\sqrt{2}$]

(*b*) If $x=1+3^{\frac{2}{3}}+3^{\frac{1}{3}}$, prove that $x^3-3x^2-6x-4=0$.

2. (*b*) Find the square root of $a+b+\sqrt{2ab+b^2}$.

$\left[\text{Ans. } \pm\frac{1}{\sqrt{2}}(\sqrt{2a+b}+\sqrt{b})\right]$

SECOND PAPER

GROUP A—Plane Geometry

2. (*a*) Same as Q. 1 (*c*) of H. S., 1961.

(*b*) ABC is a triangle, L and M are points in AB and AC respectively such that LM is parallel to BC. If $AL=5$ cms. and $LB=4$ cms., find the value of $\frac{\text{area } ALM}{\text{area } LMCB}$. $\left[\text{Ans. } \frac{25}{56}\right]$

4. (*a*) If G be the centroid of the $\triangle ABC$, show that

$$AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2).$$

(*b*) Same as Q. 2 (*b*) *or*, of H. S., 1960 (Compl.).